Game theory and Optimization Methods for Decentralized Electric Systems

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Soutenance de thèse

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Palaiseau, France.

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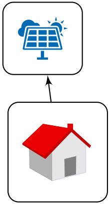

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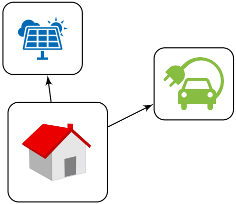

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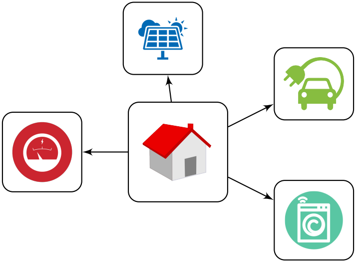

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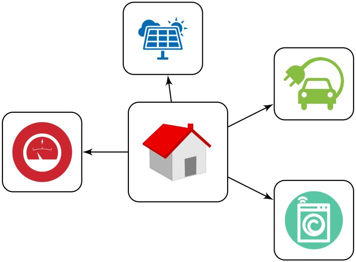

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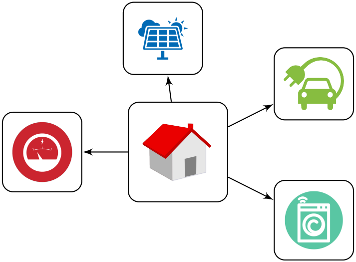

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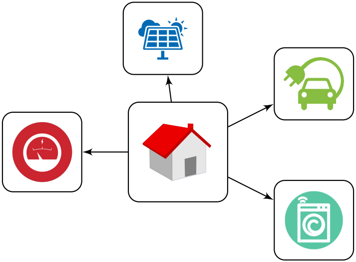
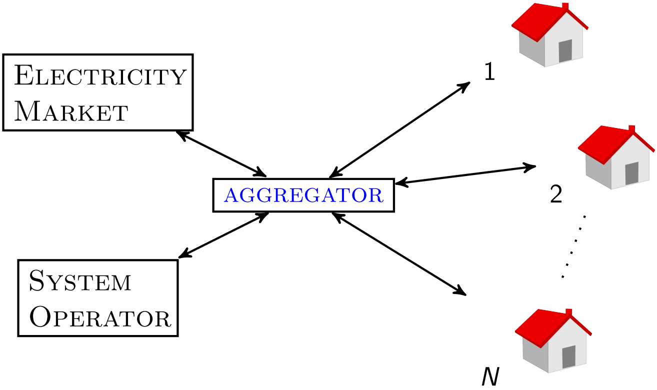


DEMAND RESPONSE: techniques to exploit consumers flexibilities

Aggregation and optimization of flexibilities

Flexibility aggregators: intermediaries between end-users and the system operator

- *aggregate* a large number of negligible flexibilities offered by end-users
- valuate them on the market or as a service offered to system operators;



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Aggregator's problem:

- large dimension;
- involving local decisions;
- decentralized and private information;

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Contributions of the thesis

Decentralized Management of Flexibilities and Optimization

1 Privacy-preserving Disaggregation for Optimal Resource Allocation

Decentralized Management of Flexibilities and Game Theory

2 Two billing mechanisms for Demand Response: Efficiency and Fairness

3 Analysis of an Hourly Billing Mechanism for Demand Response

4 Impact of Consumers Temporal Preferences in Demand Response

Efficient Estimation of Equilibria in Large Games

5 Estimation of Equilibria of Large Heterogeneous Congestion Games

6 Nonatomic Aggregative Games with Infinitely Many Types

Decentralized Energy Exchanges in a Peer to Peer Framework

7 A p2p Electricity Market Analysis based on Generalized Nash Equilibrium

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Part I

Privacy-preserving Decentralized Optimization of Flexibilities

Problem Formulation

$$\min_{\boldsymbol{x}\in\mathbb{R}^{N\times T},\ \boldsymbol{p}\in\mathbb{R}^{T}} f(\boldsymbol{p}) \tag{1a}$$

(1b)

(1c)

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How to optimize (1) **while keeping private** $(\boldsymbol{x}_n)_n$ **and** $(\mathcal{X}_n)_n$ **?**

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until DISAGGREGATION PROBLEM is feasible.

Disaggregation Feasibility

HOFFMAN Circulation's Theorem:
Disaggregation is feasible (i.e. $\boldsymbol{\mathcal{X}} \cap \mathcal{Y}_{\boldsymbol{p}} \neq \emptyset$) iff for any $\mathcal{T}_0 \subset \mathcal{T}, \mathcal{N}_0 \subset \mathcal{N}$:

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Theorem (Jacquot, Beaude, Benchimol, Gaubert, and Oudjane, 2019)

If disaggregation is not feasible, it is possible to recover a violated Hoffman cut $\mathfrak{H}_{\mathcal{T}_0, \mathcal{N}_0}$ by only local and privacy-preserving operations.

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$\boldsymbol{\mathcal{X}} = \prod_n \mathcal{X}_n$ and $\mathcal{Y} = \mathcal{Y}_{\boldsymbol{p}} = \{\boldsymbol{x} \in \mathbb{R}^{NT} \mid \sum_{n\in\mathcal{N}} \boldsymbol{x}_n = \boldsymbol{p}\}$

Require: $\boldsymbol{y}^{(0)}$, $k = 0$, ε_{cvg}, $\|.\|$

repeat

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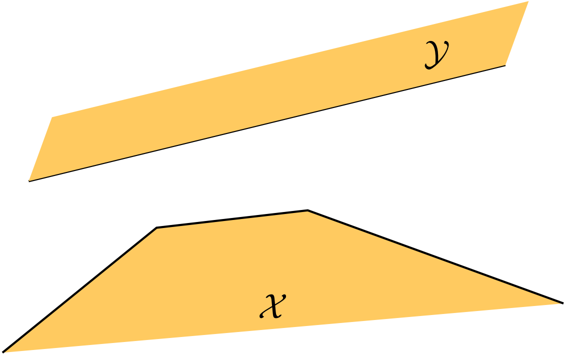

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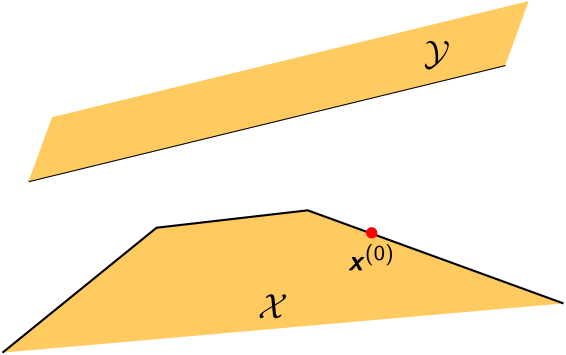

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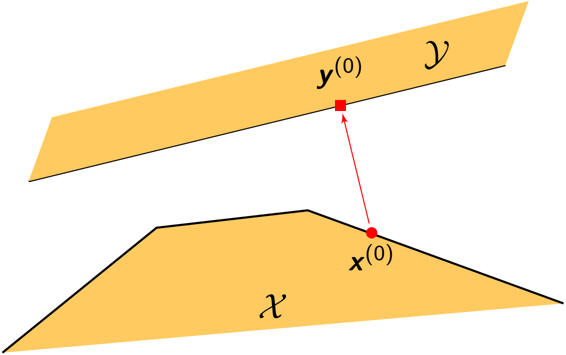

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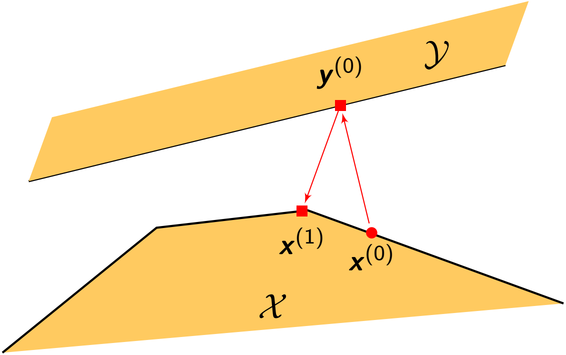

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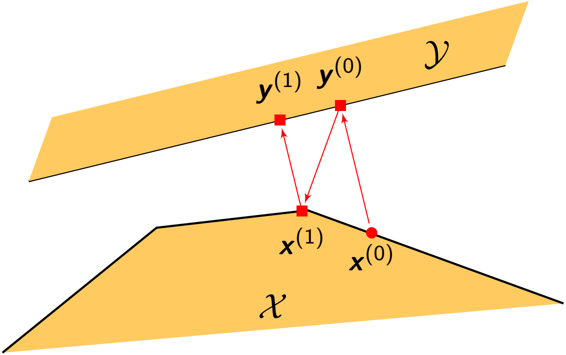

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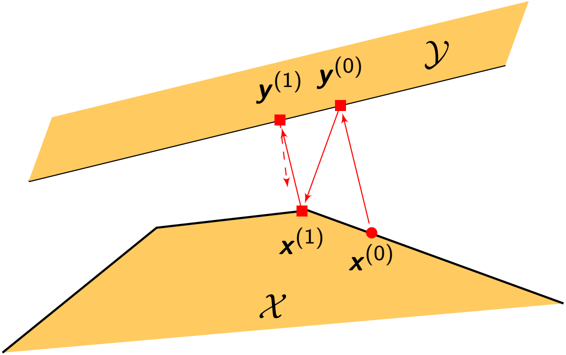

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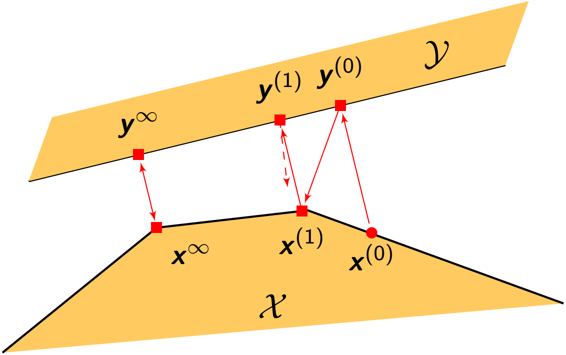
$\boldsymbol{y}^{(k+1)} \leftarrow P_{\mathcal{Y}}(\boldsymbol{x}^{(k+1)})$
$k \leftarrow k + 1$
until $\left\|\boldsymbol{y}^{(k)} - \boldsymbol{y}^{(k-1)}\right\| < \varepsilon_{\text{cvg}}$



Alternate Projections Algorithm

$\boldsymbol{\mathcal{X}} = \prod_n \mathcal{X}_n$ and $\mathcal{Y} = \mathcal{Y}_{\boldsymbol{p}} = \{\boldsymbol{x} \in \mathbb{R}^{NT} \mid \sum_{n\in\mathcal{N}} \boldsymbol{x}_n = \boldsymbol{p}\}$

Require: $\boldsymbol{y}^{(0)}$, $k = 0$, ε_{cvg}, $\|.\|$
repeat
$\boldsymbol{x}^{(k+1)} \leftarrow P_{\mathcal{X}}(\boldsymbol{y}^{(k)})$
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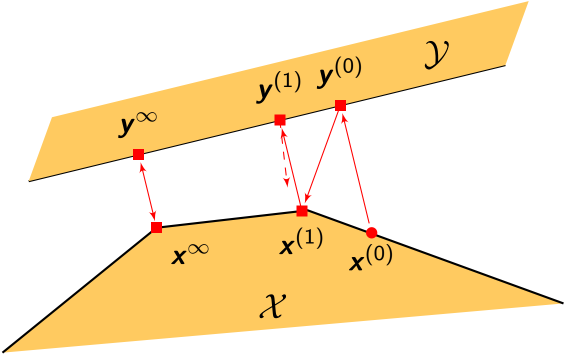

Gubin & Polyak (67): If $\mathcal{X}$ and $\mathcal{Y}$ are convex with $\mathcal{X}$ bounded, then:
$\boldsymbol{x}^{(k)} \underset{k\to\infty}{\longrightarrow} \boldsymbol{x}^{\infty} \in \mathcal{X}$ and $\boldsymbol{y}^{(k)} \underset{k\to\infty}{\longrightarrow} \boldsymbol{y}^{\infty} \in \mathcal{Y}$, with: $\|\boldsymbol{x}^{\infty} - \boldsymbol{y}^{\infty}\|_2 = \min_{\boldsymbol{x}\in\boldsymbol{\mathcal{X}},\boldsymbol{y}\in\mathcal{Y}} \|\boldsymbol{x}-\boldsymbol{y}\|_2$.

Theorem (Jacquot, Beaude, Benchimol, Gaubert, and Oudjane, 2019)

For the sets $\mathcal{X}$ and $\mathcal{Y}$ defined above, the two subsequences of APM $(\boldsymbol{x}^{(k)})_k$ and $(\boldsymbol{y}^{(k)})_k$ converge at a geometric rate to $\boldsymbol{x}^\infty \in \mathcal{X}$, $\boldsymbol{y}^\infty \in \mathcal{Y}$, with:

$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^\infty\|_2 \leq 2\|\boldsymbol{x}^{(0)} - \boldsymbol{x}^\infty\|_2 \times \left(1 - \frac{4}{N(T+1)^2(T-1)}\right)^k ,$$

and the same inequalities hold for the convergence of $\boldsymbol{y}^{(k)}$ to $\boldsymbol{y}^\infty$.

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Proof:

- rely on the notion of Friedrich angle between facets of $\mathcal{X}$ and $\mathcal{Y}$,
- consider matricial representation of these facets with positive matrices,
- then use spectral graph theory arguments to bound the cosine of the angle.

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For the sets $\boldsymbol{\mathcal{X}}$ *and* $\mathcal{Y}$ *defined above, and if* $\boldsymbol{\mathcal{X}} \cap \mathcal{Y} = \emptyset$ *, the following sets given by the limit APM orbit* $(\boldsymbol{x}^\infty, \boldsymbol{y}^\infty)$*:*

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define the cut $\mathfrak{H}_{\mathcal{T}_0, \mathcal{N}_0}$ *that is violated by* $\boldsymbol{p}$*, that is:*

$$\sum_{n \in \mathcal{N}_0} E_n + \sum_{t \in \mathcal{T}_0, n \notin \mathcal{N}_0} \overline{x}_{n,t} - \sum_{t \notin \mathcal{T}_0, n \in \mathcal{N}_0} \underline{x}_{n,t} < \sum_{t \in \mathcal{T}_0} p_t \ . \tag{3}$$

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Proposition (Jacquot, Beaude, Benchimol, Gaubert, and Oudjane, 2019)

The cut $\mathfrak{H}_{\mathcal{T}_0,\mathcal{N}_0}$ can be obtained after a finite number of APM iterations.

Benchmarks: MILP model for management of a microgrid

$$\min_{\boldsymbol{x}\in\mathbb{R}^{N\times T},\ \boldsymbol{p}\in\mathbb{R}^T} f(\boldsymbol{p})$$
$$\boldsymbol{p}\in\mathcal{P}$$
$$\sum_n x_{n,t} = p_t,\ \forall t$$
$$\boldsymbol{x}_n \in \mathcal{X}_n\ .$$

$\equiv$

$$\min_{\boldsymbol{p},\boldsymbol{p}^g,(\boldsymbol{p}^g_k),(\boldsymbol{b}_k),\boldsymbol{b}^{\mathrm{ON}},\boldsymbol{b}^{\mathrm{ST}}} \sum_{t\in\mathcal{T}} \left(\alpha_1 b^{\mathrm{ON}}_t + \sum_k c_k p^g_{k\,t} + C^{\mathrm{ST}} b^{\mathrm{ST}}_t\right)$$
$$p^g_t = \sum_{k=1}^K p^g_{k,t}\ ,\ \forall t\in\mathcal{T}$$
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$$b^{\mathrm{ST}}_t \geq b^{\mathrm{ON}}_t - b^{\mathrm{ON}}_{t-1},\ \forall t\in\{2,\dots,T\}$$
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$N =$	2^4	2^5	2^6	2^7	2^8
# master	193.6	194.1	225.5	210.9	194.0
# projs.	9507	15367	24319	26538	26646

Part II

Game Theory and Decentralized Management of Flexibilities

Why considering a Game Theory approach?

- **individual agents** (elec consumers) make consumption **decisions** based on price incentives and personal utilities,

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- **individual agents** (elec consumers) make consumption **decisions** based on price incentives and personal utilities,
- individual decisions have an **impact** on the system level,
- adopting a **decentralized** point of view: information kept locally by consumers (**privacy**).

Electricity Consumers Congestion Game

- time horizon as a finite set $\mathcal{T} = \{1, \dots, T\}$;
- set of elec consumers $\mathcal{N} = \{1, \dots, N\}$ with *flexible appliances* ;
- each $n \in \mathcal{N}$ has a feasibility set $\mathcal{X}_n$ of consumption profiles $(x_{n,t})_{t \in \mathcal{T}}$
  e.g. $\mathcal{X}_n \stackrel{\text{def}}{=} \{\boldsymbol{x}_n \in \mathbb{R}^T \mid \sum_t x_{n,t} = E_n \quad \text{and} \quad \forall t, \underline{x}_{n,t} \leq x_{n,t} \leq \overline{x}_{n,t}\}$.

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- for each t, aggregator has a per-unit energy price function $X_t \mapsto c_t(X_t)$, function of aggregated demand $X_t \stackrel{\text{def}}{=} \sum_{m\in\mathcal{N}} \boldsymbol{x}_{m,t}$ provided to consumers;

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  Examples:
  - minimize distance to target profile $(Q_t)_{t\in\mathcal{T}}$ bid on elec market
  - minimizing production costs with self production.

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- each $n \in \mathcal{N}$ minimizes the **bill**

$$b_n(\boldsymbol{x}_n, \boldsymbol{x}_{-n}) \stackrel{\text{def}}{=} \sum_{t\in\mathcal{T}} x_{n,t} c_t(X_t) \quad \text{with } \boldsymbol{x}_n \in \mathcal{X}_n;$$

Nash Equilibrium: Existence

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Assumption

For each $t \in \mathcal{T}$, $c_t(.)$ is smooth (D2), convex and strictly increasing.

Example: affine prices $\forall t \in \mathcal{T},\ c_t(x) = \alpha_t + \beta_t x$ with $\alpha_t, \beta_t \in (\mathbb{R}_+^*)^2$.

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ROSEN (65): In a game satisfying the above assumptions, there exists an NE.

Nash Equilibrium Uniqueness conditions

Proposition (Jacquot, Beaude, Gaubert, and Oudjane, 2017)

If $2|c_t'(X_t)| > \|\boldsymbol{x}_t\|_2 \, |c_t''(X_t)|$ for each $t \in \mathcal{T}$ and each feasible $\boldsymbol{x} \in \mathcal{X}$, then an NE is unique.

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Idea: generalizes Orda's result with bound constraints.

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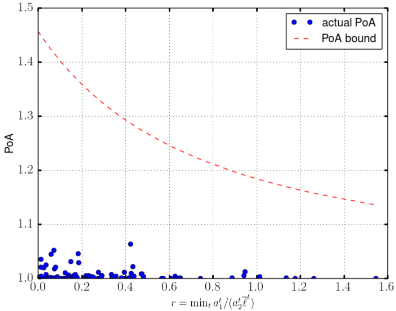
- Can have a bound for specific price parameters to ensure efficiency ?

Bounding the PoA in the affine case

Theorem (Jacquot, Beaude, Gaubert, and Oudjane, 2017)

With affine prices for each t, $c_t(X_t) = \alpha_t + \beta_t X_t$ *with* $\alpha_t \geq 0,\ \beta_t > 0$, *we have:*

$$\mathrm{PoA}(\mathcal{G}) \leq 1 + \frac{3}{4} \sup_{t \in \mathcal{T}} \left(1 + \frac{\alpha_t}{\beta_t \bar{X}_t}\right)^{-1} \xrightarrow[\frac{\alpha_t}{\beta_t \bar{X}_t} \to +\infty]{} 1$$



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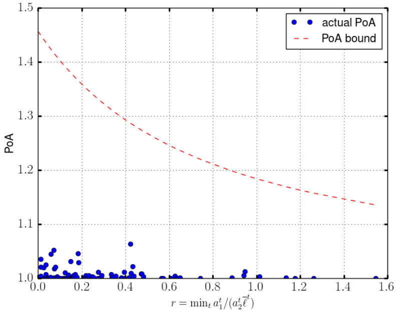

Theorem (polynomial prices, Roughgarden (2015))

If for each t, c_t is a polynomial function with positive coefficients of degree $\leq d$, then

$\mathrm{PoA}(\mathcal{G}) \leq \left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$, *and*

$\mathrm{PoA}(\mathcal{G}) \leq \frac{3}{2}$ *for affine prices.*

Two decentralized algorithms

BEST RESPONSE (BR)

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Require: $\boldsymbol{x}^{(0)}$, stopping criteria,

$k \leftarrow 0$

while not stopping criteria **do**

  for $n = 1$ to N **do**

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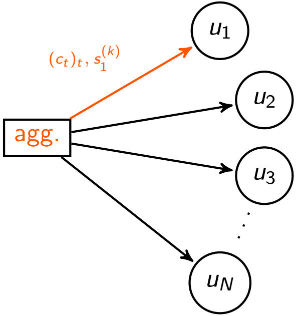

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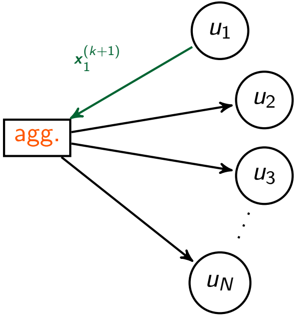

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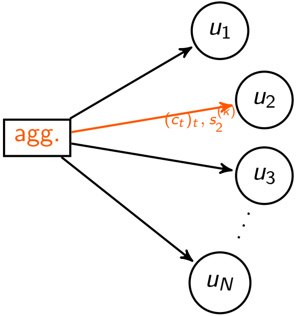

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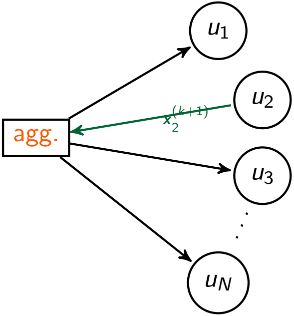

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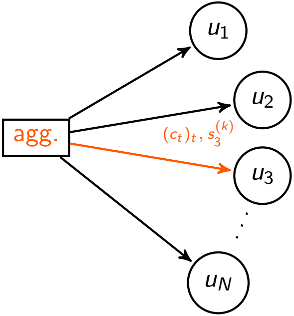

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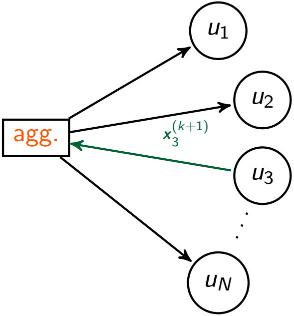

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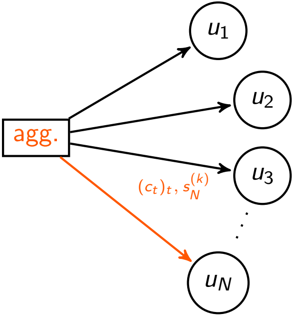

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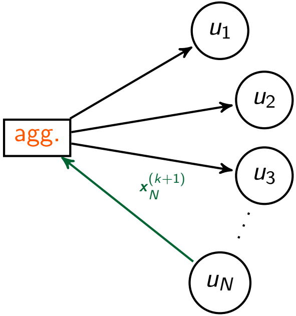

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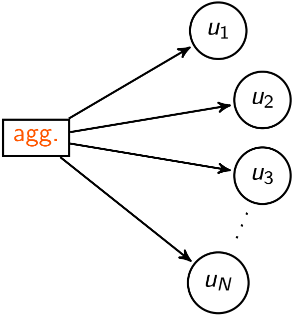

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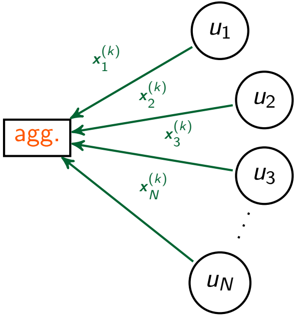

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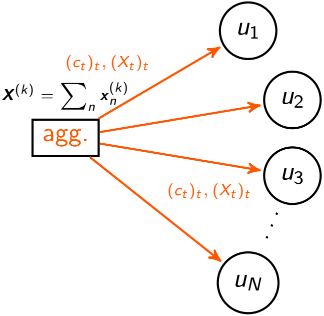

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(Fast) Convergence Results

Theorem (Jacquot, Beaude, Gaubert, and Oudjane, 2019)

With affine prices for each t, $c_t(X_t) = \alpha_t + \beta_t X_t$ with $\alpha_t \geq 0,\ \beta_t > 0$, the sequence generated by BR converge to the NE $\hat{\boldsymbol{x}}$ with:

$$\|\boldsymbol{x}^{(k)} - \hat{\boldsymbol{x}}\|_2 \leq CN \times \frac{1}{\sqrt{k}}$$

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If the operator $\hat{F}(\boldsymbol{x}) = (\nabla_{x_n} b_n(\boldsymbol{x}))_n$ is a-strongly monotone on $\mathcal{X}$, the sequence generated by SIR converge to the NE $\hat{\boldsymbol{x}}$ with:

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}^{(k)}\|_2 < (1 - \frac{a^2}{NM^2})^k \left\|\hat{\boldsymbol{x}} - \boldsymbol{x}^{(0)}\right\|_2$$

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With affine prices for each t, $c_t(X_t) = \alpha_t + \beta_t X_t$ *with* $\alpha_t \geq 0,\ \beta_t > 0$*, the sequence generated by BR converge to the NE* $\hat{\boldsymbol{x}}$ *with:*

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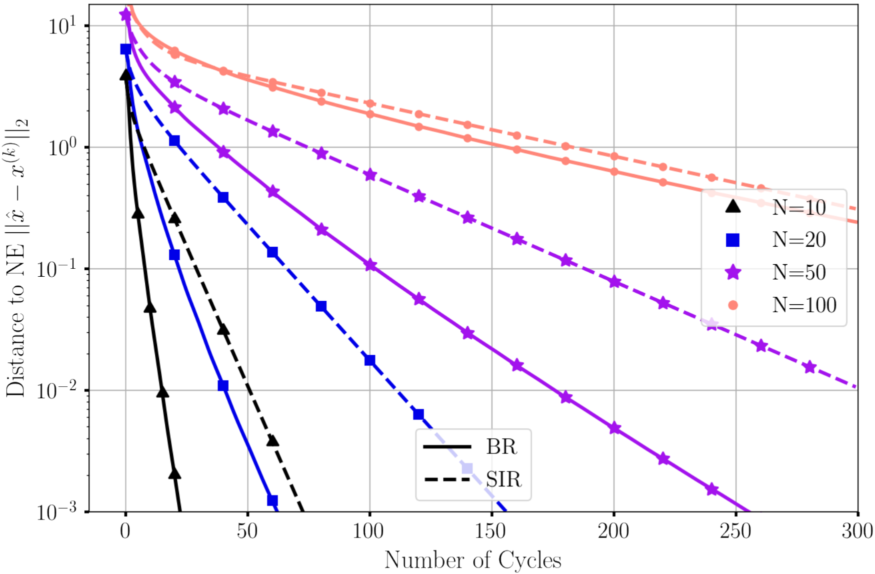
Idea: use Euclidean structure, $\nabla_n b_n$ Lipschitz and the strong monotonicity



10^1
10^0
10^{-1}
10^{-2}
10^{-3}
Distance to NE $||\hat{x} - x^{(k)}||_2$
0
50
100
150
200
250
300
Number of Cycles
N=10
N=20
N=50
N=100
BR
SIR

Online versus Offline procedure

- **Online procedure**: consider forecast updates on parameters in a stochastic environment:

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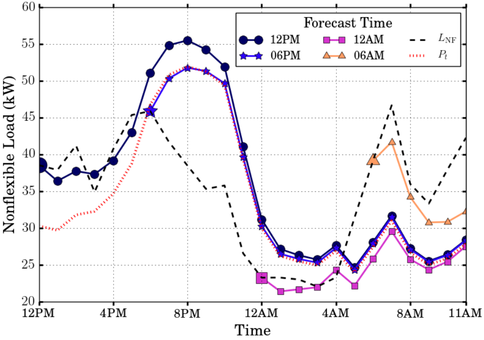

FORECASTS OF NONFLEXIBLE DEMAND

Online procedure: compute NE on “receding horizons”

Start at $t = 1$
while $t \leq T$ **do**
Set new horizon $\mathcal{T}^{(t)} = \{t, t+1, \ldots, T\}$

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  Compute NE $\boldsymbol{x}^{(t)}$ on $\mathcal{T}^{(t)}$

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  Re-compute prices $c_t(.)$ for $t \in \mathcal{T}^{(t)}$ with $\hat{\boldsymbol{D}}$
  Compute NE $\boldsymbol{x}^{(t)}$ on $\mathcal{T}^{(t)}$
  for each user $n \in \mathcal{N}$ **do**
    Realize computed profile on time t, $x^{(t)}_{n,t}$
    Update $\mathcal{X}_n^{(t+1)} \stackrel{\text{def}}{=} \left\{(x_{n,s})_{s>t} \mid (x^{(t)}_{n,t}, [x_{n,s}]_{s>t}) \in \mathcal{X}_n^{(t)}\right\}$

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  done
  Wait for $t + 1$
done

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  Wait for $t+1$
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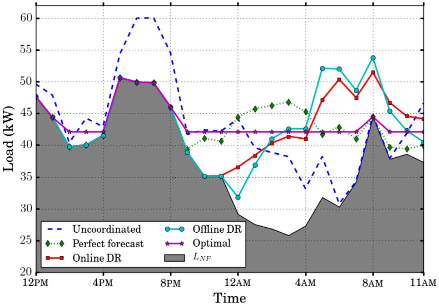
Proposition (Jacquot, Beaude, Gaubert, and Oudjane, 2019)

Under NE uniqueness and in the limit of perfect forecasts, the obtained profile $(x_{n,t}^{(t)})_{n,t}$ is an NE for the complete horizon $\{1, \dots, T\}$.

Online procedure achieves significant gains!

Cons. Scenario	Social Cost	Avg. Price	Gain
Uncoordinated	$ 1257.2	0.200 $/kWh	—
Offline DR	$ 1231.6	0.195 $/kWh	2.036%
Online DR	**$ 1131.1**	0.180 $/kWh	**10.03%**
Perfect forecast DR	$ 1075.2	0.171 $/kWh	14.47%
Optimal scenario	$ 1056.8	0.169 $/kWh	15.94%



Part III

Estimation of Equilibria of Large Heterogeneous Congestion Games

Atomic (splittable) congestion game $\mathcal{G}(\mathcal{A})$

- time horizon as a finite set $\mathcal{T} = \{1, \ldots, T\}$;
- set of agents $\mathcal{N} = \{1, \ldots, N\}$;
- each $n \in \mathcal{N}$ has a feasibility set $\mathcal{X}_n$ of (consumption) profiles $(x_{n,t})_{t \in \mathcal{T}}$;

- $\forall t$, a cost function $c_t : \mathbb{R}_+ \to \mathbb{R}$

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- $\forall n \in \mathcal{N}$, an individual utility function $u_n : \mathcal{X}_n \to \mathbb{R}$
- player n has the cost function to minimize:

$$f_n(\boldsymbol{x}_n, \overline{\boldsymbol{X}}) \stackrel{\text{def}}{=} \sum_t x_{nt} c_t(\overline{X}_t) - u_n(\boldsymbol{x}_n)$$

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- a coupling constraint set $\mathcal{A} \subset \mathbb{R}^T$ defining constraint $\overline{\boldsymbol{X}} \in \mathcal{A}$.

Nash equilibrium

Assumption (A1)

(1) $\forall t$, c_t is convex and non-decreasing on $\mathbb{R}_+$.
(2) $\forall n$, $\mathcal{X}_n$ is a convex and compact subset of $\mathbb{R}_+^T$ with nonempty relative interior.
(3) $\forall n$, u_n is concave on $\mathcal{X}_n$.
(4) $\mathcal{A}$ is a convex closed set of $\mathbb{R}^T$, and $\overline{\mathcal{X}} \cap \mathcal{A}$ is nonempty.

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Differentiable case:

Definition (NE (No coupling constraint $\Leftrightarrow \mathcal{A} = \mathbb{R}^T$))

Action profile $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$ is a Nash equilibrium (NE) if:

$$\forall n \in \mathcal{N},\ f_n(\boldsymbol{x}_n, \tfrac{1}{N}\boldsymbol{x}_n + \overline{\boldsymbol{X}}_{-n}) \leq f_n(\boldsymbol{y}_n, \tfrac{1}{N}\boldsymbol{x}_n + \overline{\boldsymbol{X}}_{-n}), \quad \forall \boldsymbol{y}_n \in \mathcal{X}_n$$

$$\Longleftrightarrow \langle \hat{F}(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0, \ \forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}},$$

$$\text{with } [\hat{F}(\boldsymbol{x})]_n \stackrel{\text{def}}{=} \nabla_{\boldsymbol{x}_n} f_n(\boldsymbol{x}_n, \overline{\boldsymbol{X}}) = \boldsymbol{c}(\overline{\boldsymbol{X}}) + (\tfrac{x_{nt}}{N} c_t'(\overline{X}_t))_t - \nabla u_n(\boldsymbol{x}_n)$$

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Differentiable case:

Definition (VNE (With coupling constraint))

Profile $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}(\mathcal{A}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \boldsymbol{\mathcal{X}} \mid \overline{\boldsymbol{X}} \in \mathcal{A}\}$ is a Variational Nash Equilibrium (VNE):

$$\langle \hat{F}(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0\,,\ \forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}}(\mathcal{A}),$$

with $$[\hat{F}(\boldsymbol{x})]_n \stackrel{\text{def}}{=} \nabla_{\boldsymbol{x}_n} f_n(\boldsymbol{x}_n, \overline{\boldsymbol{X}}) = \boldsymbol{c}(\overline{\boldsymbol{X}}) + (\tfrac{x_{nt}}{N} c_t'(\overline{X}_t))_t - \nabla u_n(\boldsymbol{x}_n)$$

Two steps of Approximation: $\mathcal{G}(\mathcal{A}) \longrightarrow \mathcal{G}^{\mathrm{na}}(\mathcal{A}) \longrightarrow \mathcal{G}^{\mathcal{I}}(\mathcal{A})$

Initial game	→	Nonatomic game
$\mathcal{G}(\mathcal{A})$		$\mathcal{G}^{\mathrm{na}}(\mathcal{A})$
VNE $\hat{\boldsymbol{x}}$ in dim $\mathbb{R}^{NT}$		SVWE $\boldsymbol{x}^*$ in dim $\mathbb{R}^{NT}$

neglect individual impact on average action $\overline{\boldsymbol{X}}$

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$\mathcal{G}(\mathcal{A})$

VNE $\hat{\boldsymbol{x}}$ in dim $\mathbb{R}^{NT}$

→

Nonatomic game

$\mathcal{G}^{\text{na}}(\mathcal{A})$

SVWE $\boldsymbol{x}^*$ in dim $\mathbb{R}^{NT}$

→

Aggregate game

$\mathcal{G}^{\mathcal{I}}(\mathcal{A})$

SVWE $\boldsymbol{x}$ in dim $\mathbb{R}^{pT}$

neglect individual impact on average action $\overline{\boldsymbol{X}}$

reduce dimension by clustering similar players

Fisrt step: Associated nonatomic game $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$ and SVWE

- each atomic player $n \in \mathcal{N}$ of $\mathcal{G}(\mathcal{A})$ $\rightarrow$ population of nonatomic players in $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$

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- symmetric action profiles: in $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$, all players in each population n play the same action $\boldsymbol{x}_n$.

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Definition

Action profile $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}(\mathcal{A}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \boldsymbol{\mathcal{X}} \mid \overline{\boldsymbol{X}} \in \mathcal{A}\}$ is a symmetric variational Wardrop equilibrium (SVWE) if

$$\langle F(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0\,,\ \forall \boldsymbol{y} \in \mathcal{X}(\mathcal{A})$$

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$$[F(\boldsymbol{x})]_n \stackrel{\text{def}}{=} \nabla_1 f_n(\boldsymbol{x}_n, \overline{\boldsymbol{X}}) \;=\; \boldsymbol{c}(\overline{\boldsymbol{X}}) - \nabla u_n(\boldsymbol{x}_n)$$

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Proposition (Existence of VNE and SVWE)

Under A1, $\mathcal{G}(\mathcal{A})$ (resp. $\mathcal{G}^{\text{na}}(\mathcal{A})$) admits a VNE (resp. SVWE).

First step of approximation: $\mathcal{G}(\mathcal{A}) \longrightarrow \mathcal{G}^{\mathrm{na}}(\mathcal{A})$

Theorem (Jacquot, Wan, Beaude, and Oudjane, 2018)

Under A1, let $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}(\mathcal{A})$ *be a VNE of* $\mathcal{G}(\mathcal{A})$ *and* $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}(\mathcal{A})$ *a SVWE of* $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$*:*

1. *if for each* $n \in \mathcal{N}$*,* u_n *is a* α*-strongly concave* $(\alpha > 0)$ *then* $\boldsymbol{x}^*$ *is unique and:*

$$\|\boldsymbol{x} - \boldsymbol{x}^*\| \leq \frac{MC}{\alpha}\sqrt{\frac{T}{N}}, \quad \text{where } M \stackrel{\text{def}}{=} \max_{\substack{\boldsymbol{x} \in \overline{cv}(\cup_n \mathcal{X}_n) \\ t \in \mathcal{T}}} |x_t|; \quad C = \max_{\substack{\boldsymbol{X} \in \overline{\mathcal{X}} \\ t \in \mathcal{T}}} |c_t'(\overline{X}_t)|$$

$$\text{besides,} \quad \frac{1}{N}\sum_n \|\boldsymbol{x}_n - \boldsymbol{x}_n^*\| \leq \frac{MC}{\alpha}\frac{\sqrt{T}}{N} \quad \text{and} \quad \|\overline{\boldsymbol{X}} - \overline{\boldsymbol{X}}^*\| \leq \frac{MC}{\alpha}\frac{\sqrt{T}}{N};$$

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1. *if for each $n \in \mathcal{N}$, u_n is a α-strongly concave ($\alpha > 0$) then $\boldsymbol{x}^*$ is unique and:*

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2. *if $(c_t)_{t \in \mathcal{T}}$ is β-strongly monotone ($\beta > 0$) then $\overline{\boldsymbol{X}}^*$ is unique and:*

$$\|\overline{\boldsymbol{X}} - \overline{\boldsymbol{X}}^*\| \leq M\sqrt{\frac{2TC}{\beta N}}.$$

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Under A1, let $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}(\mathcal{A})$ be a VNE of $\mathcal{G}(\mathcal{A})$ and $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}(\mathcal{A})$ a SVWE of $\mathcal{G}^{\text{na}}(\mathcal{A})$:

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Idea: use the VI charac of VNE/SVWE ▶ difference lying in *individual impact*

Second step: Clustering of populations in $\mathcal{G}^{\mathrm{na}}(\mathcal{A}) \rightarrow \mathcal{G}^{\mathcal{I}}(\mathcal{A})$

- Regroup similar populations of $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$ (i.e. $\mathcal{X}_n \simeq \mathcal{X}_m$ and $\nabla u_n \simeq \nabla u_m$) into a set $\mathcal{I}$ of populations with small $p \overset{\mathrm{def}}{=} |\mathcal{I}|$ and $\bigcup_{i\in\mathcal{I}} \mathcal{N}_i = \mathcal{N}$ and endow each cluster $i \in \mathcal{I}$ with:
  - a common action set $\mathcal{X}_i$ (within $\overline{conv} \cup_{n\in\mathcal{N}_i} \mathcal{X}_n$)
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- similarity of players in $\mathcal{N}_i$ measured as:

for strategy sets $(\mathcal{X}_n)_{n\in\mathcal{N}_i}$

$$\overline{\delta} = \max_{i\in\mathcal{I}} \delta_i$$

where $\delta_i \stackrel{\text{def}}{=} \max_{n\in\mathcal{N}_i} d_H(\mathcal{X}_n, \mathcal{X}_i)$

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▶ symmetric profiles: all players in i play same action $\boldsymbol{x}_i$ ▶ $\overline{\boldsymbol{X}} = \frac{1}{N}\sum_{i\in\mathcal{I}} N_i \boldsymbol{x}_i$

Second step of approximation $\mathcal{G}^{\mathrm{na}}(\mathcal{A}) \rightarrow \mathcal{G}^{\mathcal{I}}(\mathcal{A})$

Theorem (Jacquot, Wan, Beaude, and Oudjane, 2018)

Under A1, consider an approximating game $\mathcal{G}^{\mathcal{I}}(\mathcal{A})$ with $\overline{\delta}$ small enough. Let $\boldsymbol{x}$ be a SVWE of $\mathcal{G}^{\mathcal{I}}(\mathcal{A})$, and $\boldsymbol{x}^*$ a SVWE of $\mathcal{G}^{\mathrm{na}}(\mathcal{A})$. Then:

1. *if $\forall n$, u_n is α-strongly concave ($\alpha > 0$), then $\boldsymbol{x} \in \mathbb{R}^{Tp}$ and $\boldsymbol{x}^* \in \mathbb{R}^{TN}$ are unique and*

$$\|\psi_{\mathcal{I}\rightarrow\mathcal{N}}(\boldsymbol{x}) - \boldsymbol{x}^*\| \leq \sqrt{N\frac{\mathrm{err}(\overline{\delta},\overline{\lambda})}{\alpha}} \quad \textit{where } \mathrm{err}(\overline{\delta},\overline{\lambda}) \stackrel{\mathrm{def}}{=} 2TM\left(3\frac{L_{\mathrm{f}}}{\rho}\overline{\delta} + \overline{\lambda}\right) \xrightarrow[\overline{\delta},\overline{\lambda}\rightarrow 0]{} 0$$

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2. *if $(c_t)_{t\in\mathcal{T}}$ is β-strongly monotone ($\beta > 0$) then $\overline{\boldsymbol{X}}$ and $\overline{\boldsymbol{X}}^*$ are unique, and:*
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Application to DR and EV smart charging

- $\mathcal{T} = \{1, \dots, T\}$, $T = 24$: from 10 PM to 9PM the day after
- electricity prices on each $t \in \mathcal{T}$: $c_t \equiv c$:
  inclining block-rates (IBR) tariffs: pcw affine and convex functions:

$$c(\overline{X}) = \begin{cases} 1 + 200\overline{X} & \text{if } \overline{X} \leq 0.25\,, \\ -49 + 400\overline{X} & \text{if } 0.25 \leq \overline{X} \leq 0.25\,, \\ -349 + 1000\overline{X} & \text{if } 0.5 \leq \overline{X}\ . \end{cases}$$

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- $N = 2000$ consumers
  - demand constraints: $\mathcal{X}_n = \{\ \boldsymbol{x}_n \in \mathbb{R}_+^T : \sum_t x_{nt} = E_n$ and $\underline{x}_{nt} \leq x_{nt} \leq \overline{x}_{nt}\}$
    where E_n: total energy needed by n and $\underline{x}_{nt}, \overline{x}_{nt}$ are (physical) power bounds ;
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- Coupling constraints on average demand $\overline{\boldsymbol{X}}$: $\begin{array}{l} \overline{X}_t \leq 0.7, \quad \forall t \\ -0.025 \leq \overline{X}_T - \overline{X}_1 \leq 0.025 \end{array}$.

- simul number of clusters $p \in \{5, 10, 20, 50, 100\}$
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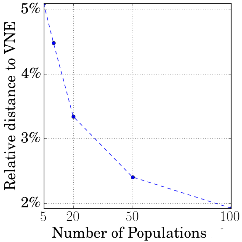
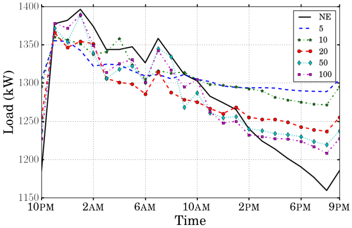


Relative error to actual VNE



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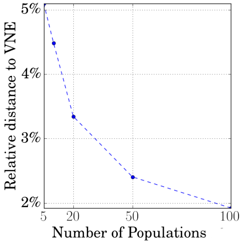

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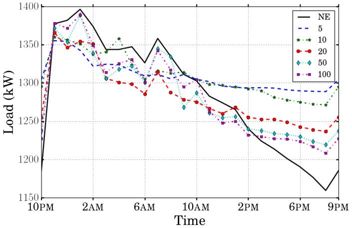

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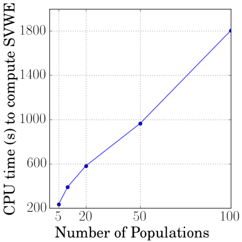

Time to compute SVWE.

- Time to compute a VNE of $\mathcal{G}(\mathcal{A})$ with the same stopping criterion: 3 h 26"
  $\rightarrow$ six times longer than the CPU time to compute the SVWE with $p = 100$.

Conclusion

Various contributions on **Distributed Optimization** and **Game Theory** for **Decentralized Electric Systems** and **Demand Response**:

- models analysis, theoretical and numerical results:

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Decentralized Management of Flexibilities and Optimization

Jacquot, P., Beaude, O., Benchimol, P., Gaubert, S., and Oudjane, N. (2019a). "A Privacy-preserving Disaggregation Algorithm for Non-intrusive Management of Flexible Energy". In: IEEE 58th Conference on Decision and Control (CDC). IEEE.

Jacquot, P., Beaude, O., Benchimol, P., Gaubert, S., and Oudjane, N. (2019b). "A Privacy-preserving Method to optimize distributed resource allocation". In: arXiv preprint.

Jacquot, P., Oudjane, N., Beaude, O., Benchimol, P., and Gaubert, S. (2018). "Procédé de gestion décentralisée de consommation électrique non-intrusif". French Patent FR1872553. EDF and Inria. filed to INPI on 7 Dec. 2018.

Decentralized Management of Flexibilities and Game Theory

Jacquot, P., Beaude, O., Gaubert, S., and Oudjane, N. (2017a). "Demand Response in the Smart Grid: the Impact of Consumers Temporal Preferences". In: IEEE International Conference on Smart Grid Communications (SmartGridComm). IEEE.

Jacquot, P., Beaude, O., Gaubert, S., and Oudjane, N. (2017b). "Demand Side Management in the Smart Grid: an Efficiency and Fairness Tradeoff". In: IEEE/PES 8thInnovative Smart Grid Technologies Europe (ISGT). IEEE.

Jacquot, P., Beaude, O., Gaubert, S., and Oudjane, N. (2019). "Analysis and Implementation of an Hourly Billing Mechanism for Demand Response Management". In: IEEE Transactions on Smart Grid 10.4, pp. 4265–4278. ISSN: 1949-3053.

Efficient Estimation of Equilibria in Large Games

Jacquot, P. and Wan, C. (2018). "Routing Game on Parallel Networks: the Convergence of Atomic to Nonatomic". In: IEEE 57th Conference on Decision and Control (CDC).

Jacquot, P. and Wan, C. (2019). "Nonatomic Aggregative Games with Infinitely Many Types". In: arXiv preprint.

Jacquot, P., Wan, C., Beaude, O., and Oudjane, N. (2018). "Efficient Estimation of Equilibria of Large Congestion Games with Heterogeneous Players". In: arXiv preprint.

Decentralized Energy Exchanges in a Peer to Peer Framework

Le Cadre, H., Jacquot, P., Wan, C., and Alasseur, C. (2019). "Peer-to-Peer Electricity Market Analysis: From Variational to Generalized Nash Equilibrium". In: European Journal of Operational Research.

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Cheng Wan

Clémence Alasseur

Hélène Le Cadre

Pascal Benchimol

Thanks to:

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and to all of you !

Issues: transmission of profiles for projection: SMC

In APM, agents still have to provide profiles $(\boldsymbol{x}_n^{(k)})_n$

$\rightarrow$ **Secure Multiparty Computation** (SMC) principle

Require: Each agent has a profile $(\boldsymbol{x}_n)_{n \in \mathcal{N}}$

for each agent $n \in \mathcal{N}$ **do**

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  Send $(s_{n,t,m})_{t \in \mathcal{T}}$ to agent $m \in \mathcal{N}$

done

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Operator computes $\boldsymbol{S} = \sum_{n \in \mathcal{N}} \boldsymbol{\sigma}_n$

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$$x_1 = s_{1,1} + s_{1,2} + s_{1,3}$$
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$$\sum_n x_n = \sigma_1 + \sigma_2 + \sigma_3$$

Extension of NE estimation in *unsplittable* case

- VI characterization of NE in atomic unsplittable case ?

Game with resources $\mathcal{T} = \{1, \dots, T\}$ and $\forall n, \mathcal{X}_n = \{e_1, \dots, , e_{K_n}\} \subset 2^{\mathcal{T}}$,

$$\forall n,\ \forall e_n = (e_{n,t})_{t \in \mathcal{T}} \in \mathcal{X}_n,\ f_n(e_n, e_{-n}) = \sum_{t \in e_n} c_t(e) = \sum_{t \in e_n} c_t(\textstyle\sum_{m: t \in e_m} 1)$$

- consider mixed strategies $\boldsymbol{x}_n \in \triangle_{\mathcal{X}_n}$:

$\hat{\boldsymbol{x}}$ is a mixed NE iff

$$\langle GV(\hat{\boldsymbol{x}}),\ \boldsymbol{x} - \hat{\boldsymbol{x}} \rangle \geq 0,\ \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}}$$

where $GV(\boldsymbol{x}) = \big(GV_n(\boldsymbol{x}_{-n})\big)_n$, with $GV_n(\boldsymbol{x}_{-n})$ the multilinear extension of f_n:

$$[GV_n(\boldsymbol{x}_{-n})]_{e_n} \stackrel{\text{def}}{=} \sum_{e_1 \in \mathcal{X}_1} \dots \sum_{e_N \in \mathcal{X}_n} x_{e_1} \dots x_{e_{n-1}} x_{e_{n+1}} \dots x_{e_N} f_n(e_1, \dots, e_{n-1}, e_n, e_{n+1}, \dots, e_N).$$

- Coupling constraint ▶ Which signification with mixed strategies ??

Wardrop formulation: flow vs cost functions

- consider *nonatomic* aggregative game $(\Theta, (f_\theta)_\theta, (\mathcal{X}_\theta)_\theta)$

WE: $\boldsymbol{x}^*$ s.t. $\forall a.e.\theta, \quad \forall \boldsymbol{x}_\theta \in \mathcal{X}_\theta, \quad f_\theta(\boldsymbol{x}_\theta^*, \boldsymbol{X}^*) \leq f_\theta(\boldsymbol{x}_\theta, \boldsymbol{X}^*)$

congestion case: if $\mathcal{X}_\theta = \{\boldsymbol{x}_\theta \in \triangle_{T-1}\} \subset \mathbb{R}^T$ and $f_\theta(\boldsymbol{x}_\theta, \boldsymbol{X}) = \sum_t x_{\theta,t} c_t(X_t)$ then:

$$\boldsymbol{x}^* \text{is a WE } \textit{iff} \quad x_{\theta,t} > 0 \Rightarrow c_t(X_t) \leq c_s(X_s) \ \forall s \in \mathcal{T}$$

but in a arbitrary *aggretative game*, $(\boldsymbol{x}_\theta, \boldsymbol{X}) \mapsto f_\theta(\boldsymbol{x}_\theta, \boldsymbol{X})$ is not *linear* in $\boldsymbol{x}_\theta$

- optimality conditions will depend on the player's actions $\boldsymbol{x}_\theta$ and not only on flow $\boldsymbol{X}$.
- consider *linear* agg. games to keep flow formulation ?

KALAI: semi-anonymous games

Bayesian games: payoff $\mu_i(c_i, c_{-i})$ depends on the **distribution** of players that choose $c_{-i} = k$

- similarity to our model $\rightarrow$ dependency of costs on the average term $(\overline{\boldsymbol{X}}_t)_t$