# Efficiency of Game-Theoretic Energy Consumption in the Smart Grid

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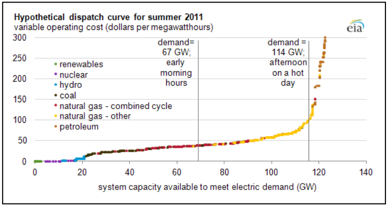
**SING13**
Paris Dauphine University

# Introduction: Cost of Flexible Consumption



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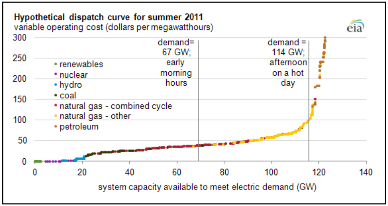

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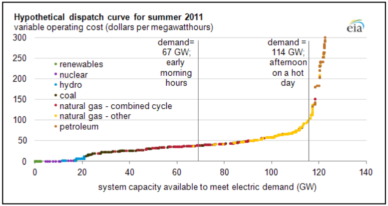


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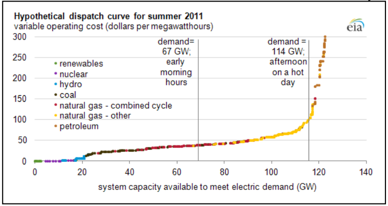


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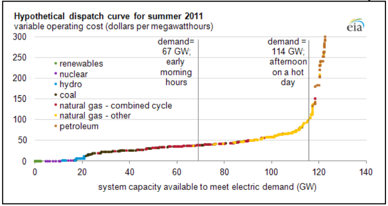


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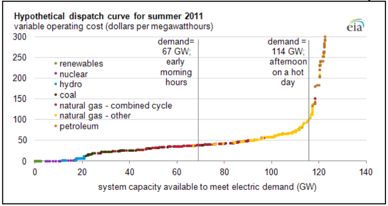


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Cost of flexible load:

$$C_t(\ell^t) := \overline{C}_t(L^t + \ell^t) - \overline{C}_t(L^t) \, .$$

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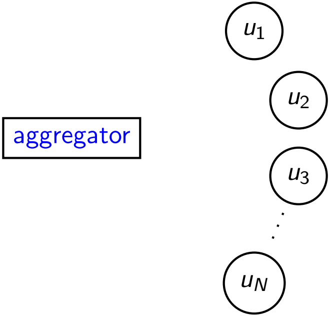

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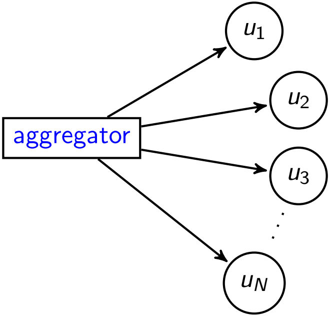


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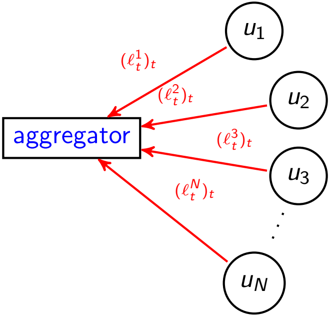


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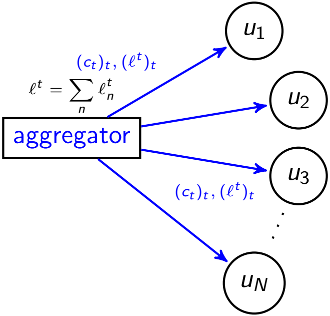


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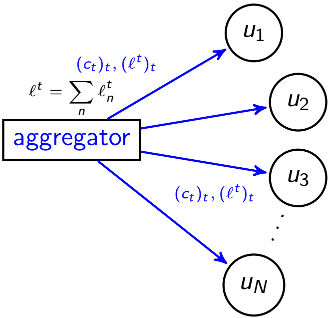


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- Consumers eventually reach an equilibrium.

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*"class B" routing game of Orda et al.*

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$\rightarrow$ The HP billing will be fairer to users.

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$(\boldsymbol{\ell}_n)_n$ is a NE *IFF* for all $n$:

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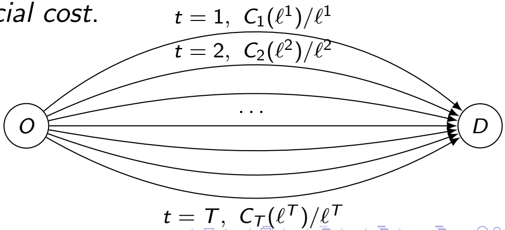

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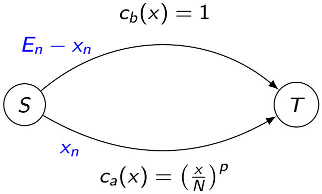


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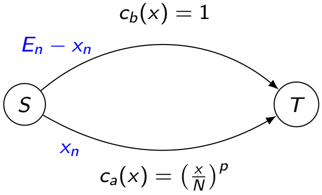


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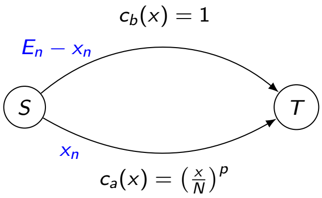


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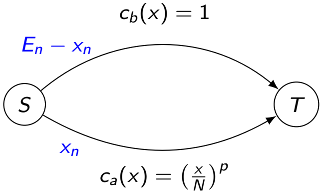


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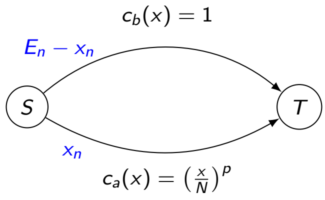


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# General Bound with Local Smoothness

## Definition (Roughgarden and Schoppmann, 2015)

**Local Smoothness.**

A cost minimization game is locally $(\lambda, \mu)$-smooth with respect to $y$ iff for all admissible outcome $x$:

$$\sum_{n=1}^{N} b_n(x_n, x_{-n}) + \nabla_{x_n} b_n(x)^T (y_n - x_n) \leq \lambda \mathrm{SC}(y) + \mu \mathrm{SC}(x).$$

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## Theorem ( Roughgarden and Schoppmann, 2015)

*If costs functions are polynomials with positive coefficients of degree $\leq d$, then $\mathrm{PoA} \leq \frac{3}{2}$ for $d = 1$ and $\mathrm{PoA} \leq \left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$ for $d \geq 2$.*

**Theorem (J. et al., 2017)**

*Assume linear prices on the arcs:*

$$c_t(\ell) = \alpha_t^t + \beta_t \ell \quad \left( = \frac{C_t(\ell)}{\ell} \right),$$

# Specific Functions: a better bound ?

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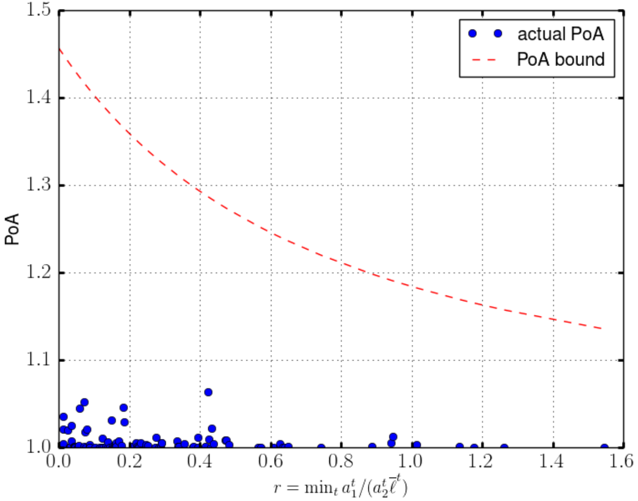
*Then the PoA is upper bounded:*

$$\begin{aligned} \mathrm{PoA} \leq \rho^{SL} &= \frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{(1+r)^2}} + \frac{1}{2(1+r)}\right) \\ &\leq 1 + \frac{3}{4}\frac{1}{1+r} \ , \end{aligned}$$

*where* $r = \inf_{t \in \mathcal{T}} \alpha^t / (\beta_t \overline{\ell}^t)$.

# Gap with Simulations



# Computation with Linear Prices and "Interior Equilibrium"

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THANK YOU!

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