Nonatomic Aggregative Games with Infinitely Many Types

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The set of feasible pure-action profiles is defined by:

$$\boldsymbol{\mathcal{X}} \triangleq \left\{ \boldsymbol{x} \in L^2([0,1], \mathbb{R}^T) \ : \ \forall \theta \in \Theta, \boldsymbol{x}_\theta \in \mathcal{X}_\theta \right\}.$$

Assumptions I

Assumption (Nonatomic pure-action sets)

The correspondence $\mathcal{X} : \Theta \rightrightarrows \mathbb{R}^T, \theta \mapsto \mathcal{X}_\theta$ *has nonempty, convex, compact values. Moreover, for all* $\theta \in \Theta$, $\mathcal{X}_\theta \subset B_R(\mathbf{0})$, *with* $R > 0$ *a constant.*

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Assumption (Measurability)

The correspondence $\mathcal{X} : \Theta \rightrightarrows \mathbb{R}^T, \theta \mapsto \mathcal{X}_\theta$ *has a measurable graph* $Gr_{\mathcal{X}} = \{(\theta, \boldsymbol{x}_\theta) \in \mathbb{R}^{T+1} : \theta \in \Theta, \boldsymbol{x}_\theta \in \mathcal{X}_\theta\}$, *i.e.* $Gr_{\mathcal{X}}$ *is a Borel subset of* $\mathbb{R}^{T+1}$.
The function $Gr_{\mathcal{X}} \to \mathbb{R}^T : (\theta, \boldsymbol{x}_\theta) \mapsto f_\theta(\boldsymbol{x}_\theta, \boldsymbol{Y})$ *is measurable for each* $\boldsymbol{Y} \in \mathbb{R}^T$.

Assumptions II

Assumption (Nonatomic convex cost functions)

For all θ, f_θ *is defined on* $(\mathcal{M}')^2$ *with* $\mathcal{M}'$ *neighborhood of* $\mathcal{M} \triangleq [0, R+1]^T$*, and:*

i) for each $\theta \in \Theta$*, function* f_θ *is continuous. In particular,* f_θ *is bounded on* $\mathcal{M}^2$*;*

ii) $\forall \theta \in \Theta$, $\forall \boldsymbol{Y} \in \mathcal{M}$, $\boldsymbol{x} \mapsto f_\theta(\boldsymbol{x}, \boldsymbol{Y})$ *is differentiable and convex on* $\mathcal{M}'$*;*

iii) there is $L_{\mathbf{f}} > 0$ *such that* $\forall \theta \in \Theta$, $\forall \boldsymbol{x}_\theta \in \mathcal{M}$, $\forall \boldsymbol{Y} \in \mathcal{M}$, $\|\nabla_1 f_\theta(\boldsymbol{x}_\theta, \boldsymbol{Y})\| \leq L_{\mathbf{f}}$ *.*

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Assumption

For each $\theta \in \Theta$ *and each* $\boldsymbol{x}_\theta \in \mathcal{M}$*, the function* $\boldsymbol{Y} \mapsto \nabla_1 f_\theta(\boldsymbol{x}_\theta, \boldsymbol{Y})$ *is continuous on* $\mathcal{M}$.

Definition (Wardrop Equilibrium (WE), [Wardrop(1952)])

A pure-action profile $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ is a pure *Wardrop equilibrium* of nonatomic aggregative game G if we have, with $\boldsymbol{X}^* = \int_{\theta\in\Theta} \boldsymbol{x}^*_\theta \,\mathrm{d}\theta$:

$$f_\theta(\boldsymbol{x}^*_\theta, \boldsymbol{X}^*) \leq f_\theta(\boldsymbol{x}_\theta, \boldsymbol{X}^*), \quad \forall \boldsymbol{x}_\theta \in \mathcal{X}_\theta, \ \forall \, a.e. \, \theta \in \Theta \ .$$

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Theorem (IDVI formulation of WE)

Under Assumptions 1 to 3, $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ is a WE of nonatomic aggregative game G if and only if either of the following two equivalent conditions is true:

$$\forall \, a.e. \, \theta \in \Theta, \quad \langle \nabla_1 f_\theta(\boldsymbol{x}^*_\theta, \boldsymbol{X}^*), \boldsymbol{x}_\theta - \boldsymbol{x}^*_\theta \rangle \geq 0 \, , \quad \forall \boldsymbol{x}_\theta \in \mathcal{X}_\theta \ , \tag{1a}$$

$$\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta), \boldsymbol{x}_\theta - \boldsymbol{x}^*_\theta \rangle \, \mathrm{d}\theta \geq 0 \, , \quad \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \ . \tag{1b}$$

Theorem (Existence of a WE, [Rath(1992)])

Under Assumption 1, Assumption 2 and Assumption 3.i), if for all θ and all $\boldsymbol{Y} \in \mathcal{M}$, $f_\theta(\cdot, \boldsymbol{Y})$ is continuous on $\mathcal{M}$, then the nonatomic aggregative game G admits a WE.

Definition (Monotone aggregative game)

With notation $\mathbf{g}_{\boldsymbol{x}}(\theta) = \nabla_1 f_\theta(\boldsymbol{x}_\theta, \int \boldsymbol{x})$, for any $\theta \in \Theta$ and any $\boldsymbol{x}, \boldsymbol{y} \in L^2([0,1], \mathcal{M})$, we say that the nonatomic aggregative game G is

i) *monotone* if $\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}}(\theta) - \mathbf{g}_{\boldsymbol{y}}(\theta), \boldsymbol{x}_\theta - \boldsymbol{y}_\theta \rangle \, \mathrm{d}\theta \geq 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in L^2([0,1], \mathcal{M})$.

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iv) *strongly monotone* with modulus α if

$$\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}}(\theta) - \mathbf{g}_{\boldsymbol{y}}(\theta), \boldsymbol{x}_\theta - \boldsymbol{y}_\theta \rangle \, \mathrm{d}\theta \geq \alpha \|\boldsymbol{x} - \boldsymbol{y}\|_2^2, \ \forall \boldsymbol{x}, \boldsymbol{y} \in L^2([0,1], \mathcal{M}) \ .$$

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v) *aggregatively strongly monotone* with modulus β if

$$\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}}(\theta) - \mathbf{g}_{\boldsymbol{y}}(\theta), \boldsymbol{x}_\theta - \boldsymbol{y}_\theta \rangle \, \mathrm{d}\theta \geq \beta \| \int \boldsymbol{x} - \int \boldsymbol{y}\|^2, \ \forall \boldsymbol{x}, \boldsymbol{y} \in L^2([0,1], \mathcal{M}) \ .$$

Example of public products game

Cost functions are given for each $\theta \in \Theta$ as:

$$f_\theta(\boldsymbol{x}_\theta, \boldsymbol{X}) = \langle \boldsymbol{x}_\theta, \boldsymbol{c}(\boldsymbol{X}) \rangle - u_\theta(\boldsymbol{x}_\theta) \ ,$$

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- Player θ's cost associated to these products is scaled by her contribution $\boldsymbol{x}_\theta$,
- $u_\theta(\boldsymbol{x}_\theta)$ measures the private utility of player θ for the contribution $\boldsymbol{x}_\theta$.

Proposition (Monotonicity of public product games)

Under above assumptions, in a public products game G, if $\mathbf{c}$ is monotone on $\mathcal{M}$ and, for each θ, u_θ is a concave function on $\mathcal{M}$, then:

i) G is a monotone game.

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iii) If $\mathbf{c}$ is strictly monotone on $\mathcal{M}$, then G is aggregatively strictly monotone.

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v) If $\mathbf{c}$ *is strongly monotone on $\mathcal{M}$ with β, then G is an aggregatively strongly monotone game with modulus β.*

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We denote $\boldsymbol{\mathcal{X}}(A) \triangleq \{\boldsymbol{x} \in \boldsymbol{\mathcal{X}} : \boldsymbol{X} = \int \boldsymbol{x} \in A\}$.

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Definition (Variational Wardrop Equilibrium (VWE))

A solution to the following IDVI problem:

$$\text{Find } \boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}(A) \text{ s.t. } \int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta), \boldsymbol{x}_\theta - \boldsymbol{x}_\theta^* \rangle \, \mathrm{d}\theta \geq 0, \quad \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}}(A),$$

is called a *variational Wardrop equilibrium* of $G(A)$.

Lemma

Under the previous assumptions on $\mathcal{X}_\cdot$*:*

i) $\boldsymbol{\mathcal{X}}$ *is a nonempty, convex, closed and bounded subset of* $L^2([0,1],\mathbb{R}^T)$*;*

ii) $\boldsymbol{\mathcal{X}}(A)$ *is a nonempty, convex and closed subset of* $\boldsymbol{\mathcal{X}}$*;*

iii) $\overline{\mathcal{X}}$ *and* $A \cap \overline{\mathcal{X}}$ *are nonempty, convex and compact subsets of* $\mathbb{R}^T$*.*

Theorem (Existence of VWE)

Under the previous assumptions, if a nonatomic aggregative game with coupling constraint $G(A)$ is monotone on $\boldsymbol{\mathcal{X}}(A)$, then a VWE exists.

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ii) if $G(A)$ is aggregatively strictly monotone on $\boldsymbol{\mathcal{X}}(A)$, then all VWE of $G(A)$ have the same aggregative profile;

iii) if G (without aggreg constraint) is aggreg. strictly monotone but, for each $\theta \in \Theta$, $\boldsymbol{Y} \in \mathcal{M}$, $f_\theta(\boldsymbol{x}, \boldsymbol{Y})$ is strictly convex in $\boldsymbol{x}$, then there is at most one WE.

Symmetric VWE with a finite-type game

Consider a game with a *finite* number of I types: $\{\mathcal{X}_\theta\}_\theta$ and $\{f_\theta\}_\theta$ are both finite.

Player set Θ divided into I measurable subsets $\Theta_1, \dots, \Theta_I$ s.t. each nonatomic player $\theta \in \Theta_i$ is of type $i \in \mathcal{I} = \{1, \dots, I\}$.

Denote common action set of players in Θ_i by $\mathcal{X}_i$ and their cost function by f_i.

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Denote common action set of players in Θ_i by $\mathcal{X}_i$ and their cost function by f_i.

Definition (Symmetric action and Symmetric VWE)

$\boldsymbol{\mathcal{X}}_S$ denotes the set of action profiles where players of same type play same action:

$$\boldsymbol{\mathcal{X}}_S \triangleq \{x \in \boldsymbol{\mathcal{X}} : \boldsymbol{x}_\theta = \boldsymbol{x}_\xi, \forall \theta, \xi \in \Theta_i, \forall i \in \mathcal{I}\} \quad \text{and } \boldsymbol{\mathcal{X}}_S(A) \triangleq \boldsymbol{\mathcal{X}}_S \cap \boldsymbol{\mathcal{X}}(A)\,.$$

A symmetric variational Wardrop equilibrium is a VWE that is symmetric.

Proposition

In a finite-type nonatomic aggregative game $G(A)$ with an aggregative constraint, a VWE is a symmetric one iff it is a solution to the following VI:

$$\text{Find } \hat{\boldsymbol{x}} \in \boldsymbol{\mathcal{X}}_S(A) \text{ s.t. } \textstyle\sum_{i\in\mathcal{I}} \langle \mathbf{g}_{\hat{\boldsymbol{x}}}(i), \mu_i \boldsymbol{x}_i - \mu_i \hat{\boldsymbol{x}}_i \rangle \geq 0,\ \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}}_S(A)\ ,$$

where μ_i is the Lebesgue measure of Θ_i.

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where μ_i is the Lebesgue measure of Θ_i.

Proposition (Existence of SVWE)

Under above assumtions, a finite-type nonatomic aggregative game $G(A)$ admits a SVWE.

Definition (Finite-type Approximating Games Sequence)

$$\{G^\nu(A^\nu) = ((\mu_i^\nu)_{i\in\mathcal{I}^\nu}, (\mathcal{X}_i^\nu)_{i\in\mathcal{I}^\nu}, (f_i^\nu)_{i\in\mathcal{I}^\nu}, A^\nu) : \nu \in \mathbb{N}^*\}$$

is a *finite-type approximating game sequence* for the game $G(A) = (\Theta, \boldsymbol{\mathcal{X}}, (f_\theta)_\theta, A)$ if $\forall \nu \in \mathbb{N}^*$, there exists a partition $(\Theta_0^\nu, \Theta_1^\nu, \dots, \Theta_{I^\nu}^\nu)$ of Θ, with $\mathcal{I}^\nu \triangleq \{1, \dots, I^\nu\}$, s.t. $\mu(\Theta_0^\nu) \triangleq \mu_0^\nu = 0$ and $\forall i \in \mathcal{I}^\nu$, $\mu(\Theta_i^\nu) \triangleq \mu_i^\nu > 0$.

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i) $\overline{\delta}^\nu \triangleq \max_{i\in\mathcal{I}^\nu} \delta_i^\nu \longrightarrow 0$, with $\delta_i^\nu \triangleq \sup_{\theta\in\Theta_i^\nu} d_H\left(\mathcal{X}_\theta, \mathcal{X}_i^\nu\right)$, and span $\mathcal{X}_i^\nu =$ span $\mathcal{X}_\theta, \ \forall \theta \in \Theta_i^\nu$.

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is a *finite-type approximating game sequence* for the game $G(A) = (\Theta, \boldsymbol{\mathcal{X}}, (f_\theta)_\theta, A)$ if $\forall \nu \in \mathbb{N}^*$, there exists a partition $(\Theta_0^\nu, \Theta_1^\nu, \dots, \Theta_{I^\nu}^\nu)$ of Θ, with $\mathcal{I}^\nu \triangleq \{1, \dots, I^\nu\}$, s.t. $\mu(\Theta_0^\nu) \triangleq \mu_0^\nu = 0$ and $\forall i \in \mathcal{I}^\nu$, $\mu(\Theta_i^\nu) \triangleq \mu_i^\nu > 0$.

players in Θ_i^ν are approximated by players of type $i \in \mathcal{I}^\nu$: as $\nu \to +\infty$:

i) $\overline{\delta}^\nu \triangleq \max_{i\in\mathcal{I}^\nu} \delta_i^\nu \longrightarrow 0$, with $\delta_i^\nu \triangleq \sup_{\theta\in\Theta_i^\nu} d_H\left(\mathcal{X}_\theta, \mathcal{X}_i^\nu\right)$, and span $\mathcal{X}_i^\nu =$ span $\mathcal{X}_\theta, \ \forall \theta \in \Theta_i^\nu$.

ii) $\overline{d}^\nu \triangleq \max_{i\in\mathcal{I}^\nu} d_i^\nu \longrightarrow 0$, where $d_i^\nu \triangleq \sup_{\theta\in\Theta_i} \sup_{(\boldsymbol{x},\boldsymbol{Y})\in\mathcal{M}^2} \left\|\nabla_1 f_i^\nu(\boldsymbol{x}_i, \boldsymbol{Y}) - \nabla_1 f_\theta(\boldsymbol{x}_\theta, \boldsymbol{Y})\right\|$.

Definition (Finite-type Approximating Games Sequence)

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iii) $D^\nu \longrightarrow 0$, where $D^\nu \triangleq d_H(A^\nu, A)$, and span $A =$ span A^ν for all $\nu \in \mathbb{N}^*$.

Theorem (Convergence of SVWE to VWE)

Under above assps, let $(G^\nu(A^\nu))_\nu$ *be a sequence of finite-type approximating games for the game* $G(A)$. *Let* $\boldsymbol{x}^*$ *be the VWE of* $G(A)$, $\hat{\boldsymbol{x}}^\nu \in \boldsymbol{\mathcal{X}}^\nu(A^\nu)$ *an SVWE of* $G^\nu(A^\nu)$ *for each* $\nu \in \mathbb{N}^*$. *Then, there exists* $\rho > 0$ *such that, with* $K_A \triangleq \frac{R+1}{\rho}$:

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$$\|\hat{\boldsymbol{X}}^\nu - \boldsymbol{X}^*\|^2 \leq \frac{1}{\beta}\Big((4L_{\mathbf{f}} + 1)K_A \max(D^\nu, \overline{\delta}^\nu) + (2M+1)\overline{d}^\nu\Big).$$

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ii) If G is strongly monotone with modulus α*, then* $(\hat{\boldsymbol{x}}^\nu)_\nu$ *, converges to* $\boldsymbol{x}^*$ *in* L^2*-norm: for all* $\nu \in \mathbb{N}^*$ *such that* $\max(\overline{\delta}^\nu, D^\nu) < \rho$,

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Without aggregate constraints, one can replace K_A *(resp.* D^ν*) by* $\frac{1}{2}$ *(resp. 0).*

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Under Assumption 1, for all $\nu \in \mathbb{N}^*$, $\|\boldsymbol{x}^\nu\|_2 \leq \overline{\delta}^\nu + R$ *for all* $\boldsymbol{x}^\nu \in \boldsymbol{\mathcal{X}}^\nu_S$.

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iv) for $i \in \mathcal{I}^\nu$, $\theta \in \Theta_i^\nu$, *and each* $\boldsymbol{x}_\theta \in \mathcal{X}_\theta$, *if* $d(\boldsymbol{x}_\theta, \mathrm{rbd}\ \mathcal{X}_\theta) > \delta_i^\nu$, *then* $\boldsymbol{x}_\theta \in \mathcal{X}_i^\nu$.

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iv) for $\boldsymbol{X} \in \text{ri}(\overline{\mathcal{X}} \cap A)$, *if* $d(\boldsymbol{X}, \text{rbd}(\overline{\mathcal{X}} \cap A)) > \max(\overline{\delta}^\nu, D^\nu)$, *then* $\boldsymbol{X} \in \overline{\mathcal{X}}^\nu \cap A^\nu$;
for $\boldsymbol{X}^\nu \in \text{ri}(\overline{\mathcal{X}}^\nu \cap A^\nu)$, *if* $d(\boldsymbol{X}^\nu, \text{rbd}(\overline{\mathcal{X}}^\nu \cap A^\nu)) > \max(\overline{\delta}^\nu, D^\nu)$, *then* $\boldsymbol{X}^\nu \in \overline{\mathcal{X}} \cap A$.

Assumption

There is a strictly positive constant η *and an action profile* $\bar{\boldsymbol{x}} \in \boldsymbol{\mathcal{X}}$ *such that, for almost all* $\theta \in \Theta$, $d(\bar{\boldsymbol{x}}_\theta, \text{rbd } \mathcal{X}_\theta) > \eta$.

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Lemma

Under Assumptions 1 and 5, there is a strictly positive constant ρ^* and a nonatomic action profile $\boldsymbol{z} \in \boldsymbol{\mathcal{X}}$ such that $\int \boldsymbol{z} \in \operatorname{ri}(\overline{\mathcal{X}} \cap A)$ and, for almost all $\theta \in \Theta$, $d(\boldsymbol{z}_\theta, \operatorname{rbd} \mathcal{X}_\theta) > 3\rho^*$.

Lemma (Convergence of $\boldsymbol{\mathcal{X}}_S^\nu(A^\nu)$ to $\boldsymbol{\mathcal{X}}(A)$)

Under Assumptions 1 and 5, let $K_A = \frac{R+1}{\rho}$*. Then, for all* $\nu \in \mathbb{N}^*$ *such that* $\max(\overline{\delta}^\nu, D^\nu) < \rho$,

- *i) for each* $\boldsymbol{x}^\nu \in \boldsymbol{\mathcal{X}}_S^\nu(A^\nu)$, $d_2(\boldsymbol{x}^\nu, \boldsymbol{\mathcal{X}}(A)) \leq 2K_A \max(D^\nu, \overline{\delta}^\nu)$;
- *ii) for each* $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}(A)$, $d_2(\psi^\nu(\boldsymbol{x}), \boldsymbol{\mathcal{X}}_S^\nu(A^\nu)) \leq 2K_A \max(D^\nu, \overline{\delta}^\nu)$.

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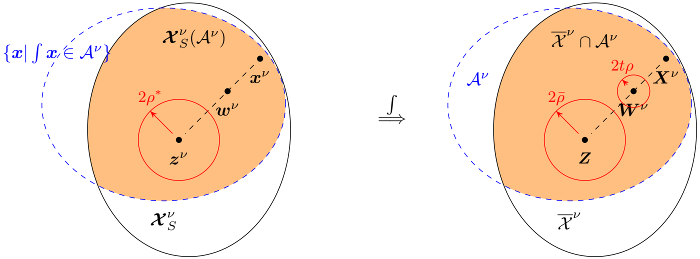

Proof of Convergence Theorem I

- Fix $\nu \in \mathbb{N}^*$, define $\hat{\boldsymbol{z}}^\nu \triangleq \Pi(\hat{\boldsymbol{x}}^\nu) \in \boldsymbol{\mathcal{X}}(A)$.

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- $\hat{\mathbf{x}}^\nu$ SVWE of $\mathcal{G}^\nu(A^\nu) \implies \int_0^1 \langle \mathbf{h}_{\hat{\mathbf{x}}^\nu}(\theta),\ \hat{\mathbf{x}}^\nu_\theta - \boldsymbol{z}^\nu_\theta \rangle \,\mathrm{d}\theta \leq 0,\ \forall \boldsymbol{z}^\nu \in \boldsymbol{\mathcal{X}}^\nu(A),$ with $\mathbf{h}_{\hat{\mathbf{x}}^\nu}(\theta) = \nabla_1 f^\nu_\theta(\hat{\mathbf{x}}^\nu_\theta, \boldsymbol{X}^\nu) = \nabla_1 f^\nu_i(\hat{\mathbf{x}}^\nu_i, \boldsymbol{X}^\nu) \triangleq \mathbf{h}_{\hat{\mathbf{x}}^\nu}(i),\ \forall \theta \in \Theta^\nu_i$ and $\forall i \in \mathcal{I}^\nu$.

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- $\forall i \in \mathcal{I}^\nu$ and $\theta \in \Theta^\nu_i$, by definition of d^ν_i , we have $\|\mathbf{h}_{\hat{\mathbf{x}}^\nu} - \mathbf{g}_{\hat{\mathbf{x}}^\nu}\|_2 \leq d^\nu_i$.

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- $\forall i \in \mathcal{I}^\nu$ and $\theta \in \Theta^\nu_i$, by definition of d^ν_i , we have $\|\mathbf{h}_{\hat{\boldsymbol{x}}^\nu} - \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}\|_2 \leq d^\nu_i$.
- $\|\hat{\boldsymbol{x}}^\nu - \hat{\boldsymbol{z}}^\nu\|_2 \leq 2K_A \max(D^\nu, \overline{\delta}^\nu)$ by preceding lemma.

With these results and $\hat{\boldsymbol{x}}_\theta^\nu \leq R + \overline{\delta}^\nu$ for all θ, one has:

$$\int_\Theta \left\langle \mathbf{g}_{\boldsymbol{x}^*}(\theta) - \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \boldsymbol{x}_\theta^* - \hat{\boldsymbol{x}}_\theta^\nu \right\rangle \mathrm{d}\theta$$

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$$\begin{aligned}
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=&\int_\Theta \left\langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \boldsymbol{x}^*_\theta - \hat{\boldsymbol{z}}^\nu_\theta \right\rangle \mathrm{d}\theta + \int_\Theta \left\langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \hat{\boldsymbol{z}}^\nu_\theta - \hat{\boldsymbol{x}}^\nu_\theta \right\rangle \mathrm{d}\theta \\
&+ \int_\Theta \left\langle \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta) - \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}^\nu_\theta - \boldsymbol{x}^*_\theta \right\rangle \mathrm{d}\theta + \int_\Theta \left\langle \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}^\nu_\theta - \boldsymbol{x}^*_\theta \right\rangle \mathrm{d}\theta
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&\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta) - \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \boldsymbol{x}_\theta^* - \hat{\boldsymbol{x}}_\theta^\nu \rangle \,\mathrm{d}\theta \\
=& \int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \boldsymbol{x}_\theta^* - \hat{\boldsymbol{z}}_\theta^\nu \rangle \,\mathrm{d}\theta + \int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \hat{\boldsymbol{z}}_\theta^\nu - \hat{\boldsymbol{x}}_\theta^\nu \rangle \,\mathrm{d}\theta \\
&+ \int_\Theta \langle \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta) - \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}_\theta^\nu - \boldsymbol{x}_\theta^* \rangle \,\mathrm{d}\theta + \int_\Theta \langle \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}_\theta^\nu - \boldsymbol{x}_\theta^* \rangle \,\mathrm{d}\theta
\end{aligned}$$

- first term is ≤ 0 (as $\boldsymbol{x}^*$ is VWE)

With these results and $\hat{\boldsymbol{x}}^{\nu}_{\theta} \leq R + \overline{\delta}^{\nu}$ for all θ, one has:

$$\begin{aligned}
&\int_{\Theta} \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta) - \mathbf{g}_{\hat{\boldsymbol{x}}^{\nu}}(\theta),\ \boldsymbol{x}^*_{\theta} - \hat{\boldsymbol{x}}^{\nu}_{\theta} \rangle \,\mathrm{d}\theta \\
=& \int_{\Theta} \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \boldsymbol{x}^*_{\theta} - \hat{\boldsymbol{z}}^{\nu}_{\theta} \rangle \,\mathrm{d}\theta + \int_{\Theta} \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \hat{\boldsymbol{z}}^{\nu}_{\theta} - \hat{\boldsymbol{x}}^{\nu}_{\theta} \rangle \,\mathrm{d}\theta \\
&+ \int_{\Theta} \langle \mathbf{g}_{\hat{\boldsymbol{x}}^{\nu}}(\theta) - \mathbf{h}_{\hat{\boldsymbol{x}}^{\nu}}(\theta),\ \hat{\boldsymbol{x}}^{\nu}_{\theta} - \boldsymbol{x}^*_{\theta} \rangle \,\mathrm{d}\theta + \int_{\Theta} \langle \mathbf{h}_{\hat{\boldsymbol{x}}^{\nu}}(\theta),\ \hat{\boldsymbol{x}}^{\nu}_{\theta} - \boldsymbol{x}^*_{\theta} \rangle \,\mathrm{d}\theta
\end{aligned}$$

- first term is ≤ 0 (as $\boldsymbol{x}^*$ is VWE)
- second term is $\leq \|\mathbf{g}_{\boldsymbol{x}^*}\|_2 \, \|\hat{\boldsymbol{z}} - \hat{\boldsymbol{x}}^{\nu}\|_2 \leq 2L_{\mathbf{f}}\, K_A \max(D^{\nu}, \overline{\delta}^{\nu})$

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&\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta) - \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \boldsymbol{x}^*_\theta - \hat{\boldsymbol{x}}^\nu_\theta \rangle \,\mathrm{d}\theta \\
= &\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \boldsymbol{x}^*_\theta - \hat{\boldsymbol{z}}^\nu_\theta \rangle \,\mathrm{d}\theta + \int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta),\ \hat{\boldsymbol{z}}^\nu_\theta - \hat{\boldsymbol{x}}^\nu_\theta \rangle \,\mathrm{d}\theta \\
&+ \int_\Theta \langle \mathbf{g}_{\hat{\boldsymbol{x}}^\nu}(\theta) - \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}^\nu_\theta - \boldsymbol{x}^*_\theta \rangle \,\mathrm{d}\theta + \int_\Theta \langle \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta),\ \hat{\boldsymbol{x}}^\nu_\theta - \boldsymbol{x}^*_\theta \rangle \,\mathrm{d}\theta
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For the last term, let $\boldsymbol{y}^{*\nu} = \psi(\boldsymbol{x}^*) \in L^2([0,1], \mathcal{M})$ and $\boldsymbol{z}^{*\nu} \triangleq \Pi^\nu(\boldsymbol{y}^{*\nu}) \in \boldsymbol{\mathcal{X}}^\nu(A^\nu)$.

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- first term is ≤ 0 (def of $\hat{\boldsymbol{x}}^\nu$ SVWE)

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- second term is $\leq (L_\mathbf{f} + \overline{d}^\nu) \, \|\boldsymbol{z}^{*\nu} - \boldsymbol{y}^{*\nu}\|_2 \leq (L_\mathbf{f} + \overline{d}^\nu) 2K_A \max(D^\nu, \overline{\delta}^\nu)$ (from def of $\overline{d}^\nu$ and lemma)

To sum up, considering ν large enough such that $\overline{d}^\nu, \overline{\delta}^\nu \leq 1$:

$$\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta) - \mathbf{h}_{\hat{\boldsymbol{x}}^\nu}(\theta), \boldsymbol{x}^*_\theta - \hat{\boldsymbol{x}}^\nu_\theta \rangle \,\mathrm{d}\theta \leq \Omega^\nu$$

$$\text{with} \quad \Omega^\nu \triangleq (4L_{\mathbf{f}} + 1)K_A \max(D^\nu, \overline{\delta}^\nu) + (2R+1)\overline{d}^\nu .$$

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Last, using the monotonicity definitions:

- if G is strongly monotone with modulus α, then $\alpha \, \|\hat{\boldsymbol{x}}^\nu - \boldsymbol{x}^*\|_2^2 \leq \Omega^\nu$;

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- if G β-is aggregatively strongly monotone , then $\beta \|\hat{\boldsymbol{X}}^{\nu} - \boldsymbol{X}^*\|^2 \leq \Omega^{\nu}$,

leading to the convergence theorem.

Case 1: Piecewise Continuous Charac – Uniform Splitting

Definition (Continuity of nonatomic player characteristic profile)

The characteristic profile $\theta \mapsto (\mathcal{X}_\theta, \nabla_1 f_\theta)$ in nonatomic aggregative game G is *continuous* at $\theta \in \Theta$ if, for all $\varepsilon > 0$, there exists $\eta > 0$ such that: for each $\theta' \in \Theta$

$$|\theta - \theta'| \leq \eta \ \Rightarrow \begin{cases} d_H(\mathcal{X}_\theta, \mathcal{X}_{\theta'}) \leq \varepsilon \\ \sup_{(\boldsymbol{x},\boldsymbol{Y}) \in \mathcal{M} \times \mathcal{M}} \|\nabla_1 f_\theta(\boldsymbol{x}, \boldsymbol{Y}) - \nabla_1 f_{\theta'}(\boldsymbol{x}, \boldsymbol{Y})\| \leq \varepsilon \ . \end{cases} \tag{2}$$

If this holds for all θ and θ' on an interval $\Theta' \subset \Theta$, then the player characteristic profile is *uniformly continuous* on Θ'.

Assume that the player characteristic profile $\theta \mapsto (\mathcal{X}_\theta, \nabla_1 f_\theta)$ of nonatomic aggregative game G is piecewise continuous, with a finite number K of discontinuity points

$$\sigma_0 = 0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_K \leq \sigma_K = 1 ,$$

and that it is uniformly continuous on (σ_k, σ_{k+1}), for each $k \in \{0, \ldots, K-1\}$.

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and that it is uniformly continuous on (σ_k, σ_{k+1}), for each $k \in \{0, \ldots, K-1\}$. For $\nu \in \mathbb{N}^*$, define an ordered set of I_ν cutting points by

$$\{\upsilon_i^\nu, i = 0, \ldots, I^\nu\} := \left\{\frac{k}{\nu}\right\}_{0 \leq k \leq \nu} \cup \{\sigma_k\}_{1 \leq k \leq K}$$

and the corresponding partition $(\Theta_i^\nu)_{i \in \mathcal{I}^\nu}$ of Θ by:

$$\Theta_i^\nu = [\upsilon_{i-1}^\nu, \upsilon_i^\nu) \text{ for } i \in \{1, \ldots, I^\nu - 1\} \ ; \quad \Theta_{I^\nu}^\nu = [\upsilon_{I_\nu - 1}^\nu, 1].$$

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Hence, $\mu_i^\nu = v_i^\nu - v_{i-1}^\nu$. Denote $\bar{v}_i^\nu = \frac{v_{i-1}^\nu + v_i^\nu}{2}$.

Proposition

Let Assumptions 1 to 4 hold, and assume that $\{\text{span } \mathcal{X}_\theta\}_{\theta\in\Theta}$ *has a finite number of elements. For* $\nu \in \mathbb{N}^*$*, consider the finite-type game* $G^\nu(A^\nu)$ *with aggregative constraint* $A^\nu \triangleq A$*, set of types* $\mathcal{I}^\nu \triangleq \{1 \dots I^\nu\}$*, where for each type* $i \in \mathcal{I}^\nu$*:*

$$\mathcal{X}_i^\nu \triangleq \mathcal{X}_{\bar{\upsilon}_i^\nu} \quad \text{and} \quad f_i^\nu(\boldsymbol{x}, \boldsymbol{Y}) \triangleq f_{\bar{\upsilon}_i^\nu}\Big(\boldsymbol{x}, \boldsymbol{Y}\Big), \ \forall(\boldsymbol{x}, \boldsymbol{Y}) \in \mathcal{M} \times \mathcal{M}.$$

Then $\left(G^\nu(A)\right)_\nu = \left(\mathcal{I}^\nu, \boldsymbol{\mathcal{X}}^\nu, A, (f_i^\nu)_{i\in\mathcal{I}^\nu}\right)_\nu$ *is a sequence of finite-type approximating games of nonatomic aggregative game* $G(A)$*.*

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i) Given $\varepsilon > 0$, there is $\eta > 0$ modulus of continuity for $\mathcal{X}_\cdot$ on (σ_k, σ_{k+1}).
For ν large enough, one has $\forall i \in \mathcal{I}^\nu$, $\mu_i^\nu < \eta$ so that $\forall \theta \in \Theta_i^\nu$, $|\bar{\upsilon}_i^\nu - \theta| < \eta$;
hence $d_H(\mathcal{X}_\theta, \mathcal{X}_i^\nu) = d_H(\mathcal{X}_\theta, \mathcal{X}_{\bar{\upsilon}_i^\nu}) < \varepsilon$.

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hence $d_H(\mathcal{X}_\theta, \mathcal{X}_i^\nu) = d_H(\mathcal{X}_\theta, \mathcal{X}_{\bar{\upsilon}_i^\nu}) < \varepsilon$.
ii) According to the continuity property, for all $(\boldsymbol{x}, \boldsymbol{Y}) \in \mathcal{M}^2$:

$$\|\nabla_1 f_i^\nu\left(\mu_i^\nu \boldsymbol{x}, \boldsymbol{Y}\right) \ - \ \nabla_1 f_\theta(\boldsymbol{x}, \boldsymbol{Y})\| = \left\|\nabla_1 f_{\bar{\upsilon}_i^\nu}\Big(\boldsymbol{x}, \boldsymbol{Y}\Big) \ - \ \nabla_1 f_\theta(\boldsymbol{x}, \boldsymbol{Y})\right\| < \varepsilon.$$

(ensure span $\mathcal{X}_\theta$ to be the same for all $\theta \in \Theta_i^\nu$: further divide if necessary)

Case 2: Finite-dim Parameterized Charac – Meshgrid

Assume that game G satisfy two conditions:

(i) action sets are K-dimensional polytopes: $\exists \boldsymbol{P} \in \mathcal{M}_{K,T}(\mathbb{R})$, and a bounded mapping $\boldsymbol{b} : \Theta \to \mathbb{R}^K$, such that for any θ,

$$\mathcal{X}_\theta = \{\boldsymbol{x} \in \mathbb{R}^T : \boldsymbol{P}\boldsymbol{x} \leq \boldsymbol{b}_\theta\},$$

which is a nonempty, compact, convex polytope.

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which is a nonempty, compact, convex polytope.

(ii) There is a bounded mapping $\boldsymbol{s} : \Theta \to \mathbb{R}^I$ such that for any $\theta \in \Theta$,

$$f_\theta(\cdot,\cdot) = f(\cdot,\cdot\,; \boldsymbol{s}_\theta)\ .$$

Furthermore, $\forall(\boldsymbol{x}, \boldsymbol{Y}) \in \mathcal{M}^2$, $\nabla_1 f(\boldsymbol{x}, \boldsymbol{Y}; \cdot)$ is Lipschitz-continuous in $\boldsymbol{s}$ with a Lipschitz constant L_3, independent of $\boldsymbol{x}$ and $\boldsymbol{Y}$.

Characteristics of player θ are parameterized by point

$$(\boldsymbol{b}_\theta, \boldsymbol{s}_\theta) \in \prod_{k=1}^{K} [\underline{b}_k, \overline{b}_k] \times \prod_{k=1}^{L} [\underline{s}_k, \overline{s}_k] \quad \subset \mathbb{R}^{K+L},$$

with $\underline{b}_k = \min_\theta b_{\theta,k}$, $\overline{b}_k = \max_\theta b_{\theta,k}$ for $k \in \{1 \dots K\}$
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For $\nu \in \mathbb{N}^*$, consider a partition of $\prod_{k=1}^{K}[\underline{b}_k, \overline{b}_k] \times \prod_{k=1}^{L}[\underline{s}_k, \overline{s}_k]$ into $I^\nu \triangleq \nu^{K+L}$ equal-sized subsets, obtained by dividing each dimension into ν equal parts.

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cutting points: $\underline{b}_{k,n_k} \triangleq \underline{b}_k + \frac{n_k}{\nu}(\overline{b}_k - \underline{b}_k)$ for $k \in \{1, \dots, K\}$,
and $\underline{s}_{k,n_k} \triangleq \underline{s}_k + \frac{n_k}{\nu}(\overline{s}_k - \underline{s}_k)$ for $k \in \{1, \dots, L\}$, with $n_k \in \{0, \dots, \nu\}$.

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Let the set of *vectorial* indices

$$\Gamma^\nu \triangleq \{\boldsymbol{n} = (n_k)_{k=1}^{K+l} \in \mathbb{N}^{K+L} \,|\, n_k \in \{1, \dots, \nu\}\} .$$

Define the partition $\Theta = \dot{\bigcup}_{\boldsymbol{n} \in \Gamma^\nu} \Theta^\nu_{\boldsymbol{n}}$ with :
$\Theta^\nu_{\boldsymbol{n}} \triangleq \left\{\theta \in \Theta : b_{\theta,k} \in [\underline{b}_{k,n_k-1}, \underline{b}_{k,n_k}) \text{ for } 1 \leq k \leq K;\ s_{\theta,k} \in [\underline{s}_{k,n_k-1}, \underline{s}_{k,n_k}) \text{ for } 1 \leq k \leq L\right\}$.

Proposition

For $\nu \in \mathbb{N}^*$, let the finite-type game $G^\nu(A^\nu)$ with an aggreg. constraint $A^\nu \triangleq A$, set of types $\mathcal{I}^\nu \triangleq \{\boldsymbol{n} \in \Gamma^\nu : \mu(\Theta^\nu_{\boldsymbol{n}}) > 0\}$ and, $\forall \boldsymbol{n} \in \mathcal{I}^\nu$,

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$$f^\nu_{\boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{Y}) \triangleq \mu^\nu_{\boldsymbol{n}} f\left(\tfrac{1}{\mu^\nu_{\boldsymbol{n}}}\boldsymbol{x}, \boldsymbol{Y}; \tfrac{1}{\mu^\nu_{\boldsymbol{n}}} \int_{\Theta^\nu_{\boldsymbol{n}}} \boldsymbol{s}_\theta \, \mathrm{d}\theta\right), \quad \forall (\boldsymbol{x}, \boldsymbol{Y}) \in \mu^\nu_i \mathcal{M} \times \mathcal{M}.$$

Then, under Assumptions 1 to 4, $(G^\nu(A))_\nu = \left(\mathcal{I}^\nu, \boldsymbol{\mathcal{X}}^\nu, A, (f^\nu_i)_{i \in \mathcal{I}^\nu}\right)_\nu$ is a sequence of finite-type approx. games of the game $G(A)$.

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i) $\overline{\delta}^\nu \to 0$: for each $\boldsymbol{n} \in \mathcal{I}^\nu$, $\mathcal{X}^\nu_{\boldsymbol{n}} = \left\{\boldsymbol{x} \in \mathbb{R}^T : \boldsymbol{P}\boldsymbol{x} \leq \frac{1}{\mu^\nu_{\boldsymbol{n}}} \int_{\Theta^\nu_{\boldsymbol{n}}} \boldsymbol{b}_\theta \, \mathrm{d}\theta\right\}$. Then, by a result generalized from [Batson(1987)],
$\exists C_0$ s.t., $\forall \theta' \in \Theta^\nu_{\boldsymbol{n}}$: $d_H\left(\mathcal{X}_{\theta'}, \mathcal{X}^\nu_{\boldsymbol{n}}\right) \leq C_0 \left\| \boldsymbol{b}_{\theta'} - \frac{1}{\mu^\nu_{\boldsymbol{n}}} \int_{\Theta^\nu_{\boldsymbol{n}}} \boldsymbol{b}_\theta \, \mathrm{d}\theta \right\| \leq \frac{C_0}{\nu} \left\| \overline{\boldsymbol{b}} - \underline{\boldsymbol{b}} \right\|$.

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ii) $\overline{d}^\nu \to 0$: for each $\boldsymbol{n} \in \mathcal{I}^\nu$ and each $\theta' \in \Theta^\nu_{\boldsymbol{n}}$, for all $(\boldsymbol{x}, \boldsymbol{Y}) \in \mathcal{M}^2$, one has:
$\|\nabla_1 f^\nu_{\boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{Y}) - \nabla_1 f_{\theta'}(\boldsymbol{x}, \boldsymbol{Y})\| = \left\|\nabla_1 f\left(\boldsymbol{x}, \boldsymbol{Y}; \frac{1}{\mu^\nu_{\boldsymbol{n}}} \int_{\Theta^\nu_{\boldsymbol{n}}} \boldsymbol{s}_\theta \, \mathrm{d}\theta\right) - \nabla_1 f(\boldsymbol{x}, \boldsymbol{Y}; \boldsymbol{s}_{\theta'})\right\| \leq \frac{L_3}{\nu} \|\overline{\boldsymbol{s}} - \underline{\boldsymbol{s}}\|$

suppose an energy operator has access to the probability distribution of the amount of flexible energy in $N = 30$ millions French households:
uniform dist. on $[0, E_{\mathrm{max}}]$ with $E_{\mathrm{max}} = 20$kWh, that is $\phi_E(E) = \frac{1}{E_{\mathrm{max}}}$ for $E \in [0, E_{\mathrm{max}}]$.

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inverse cumulative distribution function: $\forall \theta \in \Theta, \; E_\theta = F_E^{-1}(\theta) = \theta E_{\max} N$.

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G obtained is aggreg. strongly monotone with modulus $\beta = \frac{a_O}{N}$.
(G is NOT strongly monotone).

Explicit computation of aggregate VWE profile as :

$$\int_\Theta \langle \mathbf{g}_{\boldsymbol{x}^*}(\theta), \boldsymbol{x}_\theta - \boldsymbol{x}^*_\theta \rangle \,\mathrm{d}\theta \geq 0, \quad \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \iff \int_\Theta \langle \boldsymbol{c}(\boldsymbol{X}^*), \boldsymbol{x}_\theta - \boldsymbol{x}^*_\theta \rangle \,\mathrm{d}\theta \geq 0, \quad \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}}$$

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With aggregate flexible energy $E_{\mathrm{tot}} \triangleq \int_\Theta E_\theta \, \mathrm{d}\theta = \frac{1}{2} N E_{\mathrm{max}}$, we obtain:

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$\boldsymbol{X}^*$ is the solution of the quadratic program:

$$\min_{\boldsymbol{X}} \frac{a_O}{N} \times \frac{1}{2} X_O^2 + \frac{a_P}{N} \times \frac{1}{2} X_P^2$$
$$X_O + X_P = E_{\mathrm{tot}}$$
$$0 \leq X_O, X_P$$

that is: $\boldsymbol{X}^* = (X_O^*, X_P^*) = \left(\frac{a_P}{a_O + a_P} E_{\mathrm{tot}}, \frac{a_O}{a_O + a_P} E_{\mathrm{tot}} \right)$.

sequence of finite-type approximating games G^ν

for each $\nu \in \mathbb{N}^*$, $I^\nu = \nu$
population split uniformly with $\Theta_i^\nu = [\frac{i-1}{I}, \frac{i}{I}]$, for each $i \in \mathcal{I} = \{1, \dots, I\}$.
consider for each $i \in \mathcal{I}$:

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Computing the aggregate approximate equilibrium, we obtain:

$$\hat{\boldsymbol{X}}^I = \left(\tfrac{a_P}{a_O + a_P} E_{\text{tot}} (1 + \tfrac{1}{I}), \tfrac{a_O}{a_O + a_P} E_{\text{tot}} (1 + \tfrac{1}{I}) \right) = (1 + \tfrac{1}{I}) \boldsymbol{X}^* \ ,$$

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and thus we have:

$$\|\hat{\boldsymbol{X}}^I - \boldsymbol{X}^*\| = \frac{\|\boldsymbol{X}^*\|}{I} = \frac{\sqrt{a_O^2 + a_P^2}}{a_O + a_P} E_{\mathrm{tot}} \times \frac{1}{I}. \tag{4}$$

Applying the convergence theorem, with:
$L_{\mathbf{f}} = \max_{\boldsymbol{X}\in\overline{\mathcal{X}}} \|\boldsymbol{c}(\boldsymbol{X})\| = \frac{a_P}{N} E_{\text{tot}}$
we obtain:

$$\|\hat{\boldsymbol{X}}^l - \boldsymbol{X}^*\|^2 \leq \frac{1}{\beta} 2 L_{\mathbf{f}} \overline{\delta}^l = \frac{N}{a_O} 2 \frac{a_P}{N} E_{\text{tot}} \times \frac{2E_{\text{tot}}}{l}$$

$$\iff \|\hat{\boldsymbol{X}}^l - \boldsymbol{X}^*\| \leq 2E_{\text{tot}} \sqrt{\frac{a_P}{a_O}} \times \frac{1}{\sqrt{l}} \ .$$

Conclusion

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Thank you!

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