Analysis and Implementation of an Hourly Billing Mechanism for Demand Response Management

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***Abstract*—Game theory has been shown to be a valuable tool to study strategic electricity consumers enrolled in a demand response program. Among the different billing mechanisms proposed in the literature, the hourly billing model is of special interest as an intuitive and fair mechanism. We focus on this model and answer to several theoretical and practical questions. First, we prove the uniqueness of the consumption profile corresponding to the Nash equilibrium, and we analyze its efficiency by providing a bound on the Price of Anarchy. Next, we address the computational issue of this equilibrium profile by providing results on the convergence rates of two decentralized algorithms to compute the equilibrium: the cycling best response dynamics and a projected gradient descent method. Last, we simulate this demand response framework in a stochastic environment where the parameters depend on forecasts. We show numerically the relevance of an online demand response procedure which reduces the impact of inaccurate forecasts in comparison to a standard offline procedure.**

***Index Terms*—Smart Grid, Demand Response, Demand Side Management, Game Theory, Nash Equilibrium, Best Response.**

I. Introduction

DEMAND Response (DR) is a technique to exploit electricity consumers flexibilities by giving them particular incentives, in order to achieve some services to the grid e.g. reducing production, transmission and distribution costs or increasing renewable energy insertion [1]. In DR programs, the aggregated energy demand is a key metric and an aggregator interacts with active consumers—willing to minimize their electricity bill or maximize their utility—to optimize this demand profile. In such a framework, energy can be viewed as an asset demanded by customers, and which has a cost that depends on total demand and the time of demand. Some congestion effects arise on the most demanded time periods.

Various aspects of DR have been investigated in the existing literature (consumers personal utilities and discomfort related to their electricity consumption, consumers privacy, network and power flow constraints), often leading to complex optimization problems. Owing to the high number of variables (electrical appliances of consumers) and privacy concerns, the need for a decentralized optimization approach is a global consensus [2].

Among the different decentralized approaches for a DR system, several works rely on a dual decomposition of the optimization problem of a centralized entity [3–7]: in that case, the centralized entity computes the vector of Lagrange multipliers (e.g. from the supply-demand balancing constraint) of his problem and sends it to consumers as prices for each time period. An iterative algorithm between the entity (updating the Lagrange multipliers) and the consumers (adjusting their consumption) is run until convergence of the consumption profiles (decomposition-coordination).

However, this kind of approach does not capture the effect of strategic consumers willing to minimize their electricity bill. To answer this issue, different game-theoretic frameworks have been proposed in the smart grid literature, e.g. [8–13], following the seminal paper [14]. There is an interaction between consumers as their billing functions depend on the load of other consumers, usually through the aggregated load. Among the different game theory models, our paper follows the same approach than [10–13]: we consider an hourly billing mechanism, in which a consumer pays for each time period (e.g. each hour) proportionally to the energy consumed on this period. This billing mechanism has a structure of a *routing congestion game* [15] and has been shown to have important properties of fairness and incentives compatibility [13], while remaining simple and intuitive.

In game-theoretic models, a major issue is to define an effective procedure to compute and reach the consumption *equilibrium* associated with the game. Several papers [10, 11, 16] have investigated the complexity and algorithmic aspects associated to the notion of equilibrium. In [11] and [10], the authors consider the same billing mechanism as the one studied in this paper, and propose decentralized methods to compute the Nash equilibrium (an iterative proximal best response in [11] and a proximal-point algorithm in [10]).
In this paper, we investigate the theoretical properties and computational aspects of the hourly billing mechanism and discuss its practical implementation.
This paper brings five theoretical and practical contributions:

1) We give a new result on the uniqueness of the equilibrium (Thm. 2) under a convexity assumption. This result extends [10, Prop. 1]—which relies on the general results of [17]—where uniqueness is given for a particular class of price functions (of the form $c_t(\ell) = \alpha_t + \beta_t \ell^{b_t}$ with $\alpha_t \geqslant 0, \beta_t > 0$ and $b_t \geqslant 1$). In contrast, our uniqueness theorem applies to any convex and strictly increasing price functions. It extends [15, Thm. 1] to a more general model of constraints where we consider upper and lower bounds on the load at each time period;

2) We give a new result on the induced *Price of Anarchy* (PoA). This result (Thm. 3) gives an evaluation of the equilibrium efficiency in terms of social cost. The PoA is

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numerically close to one but not one. To our knowledge, there are very few existing results on the PoA for this framework. A related but different result is [12], where the authors consider a maximization game with individual utilities, and consider the same hourly proportional billing. A bound on the PoA is obtained assuming that players individual utilities are large enough compared to the system cost. Our bound applies to the minimization game without utilities, but is tighter asymptotically. Another related work is [18] where the authors prove that the PoA converges to one in the asymptotic case of an infinite number of players;

3) We bound the convergence rate of the Best Response (BR) algorithm in the case of affine prices (Thm. 4). In that case, convergence is known but, to our knowledge, no bound on the rate has ever been given. The convergence has been conjectured more generally for any convex prices [19, 20].

4) We introduce a different algorithm: SIRD, based on a simultaneous projected gradient descent (Algo 2), and show its geometric convergence (Thm. 5) with a condition on the price functions only. To our knowledge, those results are also new. The proposed algorithms (BR and SIRD) and their convergence rates are compared numerically with the algorithms proposed in [10] and [11]. In the case of SIRD, we allow a fix step-size and we do not need a proximal term so the convergence is faster;

5) Last, we introduce an online DR procedure with receding horizons (Algo 3), in the spirit of Model Predictive Control [21], to take into account updated forecasts in a stochastic environment. We prove that the consumption profiles computed by this procedure correspond to the desired NE in the limit of perfect forecasts (Thm. 6). We show numerically, based on real consumption data, that this procedure can achieve significant savings compared to an offline procedure.

This paper reassembles and extends the main results on the hourly billing model announced in our conference papers [22, 23]. Following these papers, in Thm. 3 here, we give the upper bound on the Price of Anarchy [22, Thm. 2] and in Prop. 1 we use the same property than [22, Thm. 1]. We also use the potential property of the game noticed in [23, Thm. 2] and the *Best Response* algorithm presented in [22, Def. 3]. However there are several additional results in this paper: the theorem of uniqueness of the equilibrium presented here (Thm. 2) is stronger than [22, Thm. 1]. Also, the SIRD algorithm (Algo 2) and the convergence theorems (Thms. 4 and 5) were not presented in [22, 23]. Finally, we complete the simulation framework in [22, 23] by considering updated forecasts on the nonflexible load and by introducing an online demand response procedure (Algo 3).

This paper is organized as follows: Sec. II gives the mathematical model of the DR framework and the associated billing mechanism, under the form of a game. In Sec. III, we define two decentralized algorithms that enable to compute the equilibrium consumption profiles. We prove the convergence of those algorithms and provide upper bounds on their convergence rates. We present a numerical study of the given algorithms and compare them to two others algorithms from [10] and [11]. Finally, in Sec. IV, we define an *online* DR procedure and simulate it with historical consumption data of consumers with electric vehicles as flexible consumptions. We compare the performance of this online DR scheme to the *offline* version and other consumption scenarios.

Notation Convention: through this paper, bold font $\boldsymbol{\ell}$ is used to denote a vector as opposed to a scalar ℓ.

II. Consumption Game with Hourly Billing

A. District of flexible consumers

We consider a set $\mathcal{N} = \{1, \dots, N\}$ of residential consumers linked to a local *aggregator*. Each household is equipped with a smart meter enabling two-way communication of information with the aggregator. We assume that each household electricity consumption can be divided into two parts: one which is *nonflexible* (lights, cooking appliances, TVs) and one which is *flexible* (Electric Vehicle charging, water heating, etc). Moreover, each smart meter is linked to an *Electricity Consumption Scheduler* (ECS) that can automatically optimize and schedule the consumption profile of the consumer's flexible appliances, given the constraints set by the consumer and the physical constraints of each appliance.

B. From individual to aggregated consumption profiles

In the DR program, we determine a consumption profile for each consumer on a finite time horizon T. In this study, we take $\mathcal{T}$ as a discrete set of time periods $\mathcal{T} = \{1, \dots T\}$. In the simulations, $\mathcal{T}$ will correspond to one day, and each time period t to one hour. The aggregated flexible load profile on the set of consumers is obtained as:

$$\boldsymbol{L} = (L_t)_{t\in\mathcal{T}} \in \mathbb{R}^T \;\text{ with }\; \forall t \in \mathcal{T}, L_t \stackrel{\Delta}{=} \textstyle\sum_n \ell_{n,t} \;, \qquad (1)$$

where $\ell_{n,t}$ denotes the flexible consumption of consumer n on time period t.

C. Aggregator objective from the aggregated consumption

The aggregator is himself linked to electricity providers and we consider that he faces a per-unit (of energy) price function $L_t \mapsto c_t(L_t)$ associated with each time period $t \in \mathcal{T}$ for the flexible electricity demand L_t given in (1). The total system cost for providing the flexible profile $(L_t)_{t\in\mathcal{T}}$ is then:

$$\mathcal{C}(\boldsymbol{L}) \stackrel{\Delta}{=} \sum_{t\in\mathcal{T}} L_t \times c_t(L_t) \;, \qquad (2)$$

a quantity that should be minimized by the aggregator. In particular, we make the assumption that the system cost $\mathcal{C}$ is time-separable. The prices $(c_t)_t$ can either correspond to real prices or be abstract functions revealing the objective function of the aggregator, up to an additive or multiplicative constant, as seen in the three practical examples below.

Ex. 1. *The aggregator has taken positions $(L_t^{\mathrm{DA}})_t$ on the Day-Ahead market. Then he is facing penalties on the balancing market, and wants to minimize the distance to its bid profile:*

$$\mathcal{C}(\boldsymbol{L}) = \left\| \boldsymbol{L}^{\mathrm{DA}} - \boldsymbol{L} \right\|_2^2 = \sum_{t\in\mathcal{T}} (L_t^{\mathrm{DA}})^2 + \sum_{t\in\mathcal{T}} L_t \times (2L_t^{\mathrm{DA}} + L_t) \;.$$

Ex. 2. *The aggregator owns a source of renewable energy and forecasts a production profile $(\hat{G}_t)_t$. He wants to maximize the flexible consumption when $\hat{G}_t$ is the most important [24], and can therefore minimize:*

$$\mathcal{C}(\boldsymbol{L}) = \left\| \hat{\boldsymbol{G}} - \boldsymbol{L} \right\|_2^2 = \sum_{t\in\mathcal{T}} \hat{G}_t^2 + \sum_{t\in\mathcal{T}} L_t \times (2\hat{G}_t + L_t) \ .$$

Ex. 3. *The aggregator has his own production facilities with convex and increasing production cost $C(D)$ where D is the total power to be provided. If the set of consumers has a total aggregated nonflexible profile $\boldsymbol{L}_{\mathrm{NF}}$, then at each time the total demand is $D_t = L_{\mathrm{NF},t} + L_t$. The additional cost for the flexible load is $C(L_{\mathrm{NF},t} + L_t) - C(L_{\mathrm{NF},t})$ and the aggregator will minimize:*

$$\mathcal{C}(\boldsymbol{L}) = \sum_t L_t \times \left(\frac{C(L_{\mathrm{NF},t} + L_t) - C(L_{\mathrm{NF},t})}{L_t} \right) \ ,$$

where the term between parentheses can be set as the price signal $c_t(L_t)$ to be sent to consumers for coordination.

Note that in Examples 1 and 2, the functions (c_t) are not directly related to real prices but act rather as signals for coordination between consumers.

In our framework, we will consider the following different assumptions on the price functions $(c_t)_t$.

Assumption 1. *For each $t \in \mathcal{T}$, c_t is twice differentiable, strictly increasing and convex.*

Assumption 2. *For each $t \in \mathcal{T}$, c_t is twice differentiable, convex and strictly increasing. Moreover, there exists $a > 0$ s.t. for any t and admissible $\boldsymbol{\ell}$:*

$$2c_t'(L_t) \left(1 - \left(\tfrac{c_t''(L_t)}{2c_t'(L_t)} \right)^2 \|\boldsymbol{\ell}_t\|_2^2 \right) \geqslant a. \tag{3}$$

Assumption 3. *For each $t \in \mathcal{T}$, c_t is affine, positive and increasing: $\forall t \in \mathcal{T},\ c_t(\ell) = \alpha_t + \beta_t \ell$ with $\alpha_t, \beta_t \in (\mathbb{R}_+^*)^2$.*

The three latter assumptions are more and more restrictive: Assumption 3 implies Assumption 2 with $a = 2\min_t \beta_t$, and Assumption 2 implies Assumption 1. Note that Assumption 3 provides a practical case for which all our results hold, and is verified in the case of Examples 1 and 2.

Remark 1. *For $a = 0$, inequality (3) in Assumption 2 simplifies to the condition: $\|\boldsymbol{\ell}_t\|_2^{-1} \geqslant \left| \frac{c_t''(L_t)}{2c_t'(L_t)} \right|$. For each t, c_t'' has to be small relatively to c_t'.*

Assumption 1 is standard in the congestion games literature and corresponds to "type-B" functions in the seminal paper [15]. This assumption is also made in most of the papers dealing with game-theoretic DR models as [13]. Indeed, it is justified by the fact that marginal costs of producing and providing electricity are increasing. Assumption 3 is also a standard assumption made in [25] because it enables fast computation of NE (see Sec. III), but it is restrictive, although several papers as [11, 14] simply consider linear price functions $c_t(\ell) = \beta_t \ell$. Last, Assumption 2 is not very explicit but is an in-between condition that comprises a larger set of functions than linear functions and for which our main results hold. For instance, the assumption holds for the family of polynomial functions considered in [10]: $c_t(\ell) = \alpha + \beta \ell^{\nu_t}$ with $\alpha \geqslant 0, \beta > 0$, $1 \leqslant \nu_t < 3$ and if $L_t > 0$ for each $t \in \mathcal{T}$. More generally, this condition will be verified if c_t'' is small enough compared to c_t'.

Whatever the assumption retained, the objective of the aggregator is to send the right incentives to consumers through a *billing mechanism* in order to minimize his costs. A billing mechanism does not refer to a real billing system but more generally to a *signal* sent in order to coordinate consumers. It is given as a tuple of billing functions $(b_n)_{n\in\mathcal{N}}$ chosen by the aggregator to recover the global system cost $\mathcal{C}(\boldsymbol{L}) = \sum_{t\in\mathcal{T}} L_t c_t(L_t)$. As a result, the billing functions b_n are chosen such that $\mathcal{C} = \sum_{n\in\mathcal{N}} b_n$. Of course, one can always consider a profit ratio κ if the billing functions are used to design real consumer bills (the bill of n is set to κb_n). The function b_n depends of course on n's flexible consumption profile $\boldsymbol{\ell}_n$ but also depend on the load of the other consumers through the aggregated load $\boldsymbol{L}$.

D. Consumer's Optimization Problems

In this paper, following our studies in [22, 23], we will use an hourly proportional billing mechanism, where each consumer $n \in \mathcal{N}$ minimizes her bill:

$$b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) \triangleq \sum_{t\in\mathcal{T}} \ell_{n,t} c_t(L_t) = \sum_{t\in\mathcal{T}} \ell_{n,t} c_t\big(\ell_{n,t} + s_{n,t}\big), \tag{4}$$

where $\boldsymbol{\ell}_{-n} \triangleq (\boldsymbol{\ell}_m)_{m\neq n}$ denotes the consumption of all consumers but n and $s_{n,t} \triangleq \sum_{m\neq n} \ell_{m,t}$.

This billing mechanism was shown to have interesting fairness properties and is also adequate when considering consumers' utility functions (representing, e.g., temporal preferences for flexible consumption) [22, 23, 25]. This mechanism gives a particular *aggregative* structure, where the dependency to the others is only through the aggregated load $L_t = \ell_{n,t} + s_{n,t}$.

Through her ECS, each consumer will adjust her flexible consumption profile $\boldsymbol{\ell}_n \in \mathbb{R}^T$ to minimize her bill, which corresponds to the following optimization problem:

$$\min_{\boldsymbol{\ell}_n \in \mathcal{L}_n} b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) \tag{5}$$

where $\mathcal{L}_n \subset \mathbb{R}^T$ is the set of consumer n feasible profiles. In the remaining of the paper, we assume the following:

Assumption 4. *For each $n \in \mathcal{N}$, $\mathcal{L}_n$ is compact and convex.*

Problem (5) is a convex nonlinear mathematical program for which efficient methods of resolution exist [26]. Most of the results given in this paper hold without any further assumptions than Assumption 4, but we will focus on feasibility sets of the form (6), also considered in [6, 10, 13, 14, 25, 27].

Main Example 1. *Deferrable load with fixed energy demand:*

$$\mathcal{L}_n \triangleq \Big\{ \boldsymbol{\ell}_n \in \mathbb{R}^T \ \textit{s.t.} \ \textstyle\sum_{t\in\mathcal{T}} \ell_{n,t} = E_n \ , \tag{6a}$$

$$\underline{\ell}_{n,t} \leqslant \ell_{n,t} \leqslant \overline{\ell}_{n,t}, \forall t \in \mathcal{T} \ \Big\}. \tag{6b}$$

Constraint (6a) *ensures that the total energy given to* n *satisfies her daily flexible energy demand over* $\mathcal{T}$*, denoted by* E_n*, that we assume fixed and deterministic*[1]. *Constraint* (6b) *takes into account the physical power constraints and the personal scheduling constraints (supposed given by the user to her ECS). Note that taking* $\underline{\ell}_{n,t} = \overline{\ell}_{n,t} = 0$ *forces* $\ell_{n,t} = 0$ *so that constraint* (6b) *includes in particular unavailability during some time periods. Constraints* (6) *give a simple model for deferrable loads such as water heaters (energy to heat a quantity of cold water between refill and usage periods) or electric vehicles (energy to be charged in the battery during parking period) [4].*

Note that (6) is a generalization of routing "atomic splittable" congestion games [15], well studied in the game theory literature, where the feasibility sets generally considered are $\mathcal{L}_n \stackrel{\Delta}{=} \{\boldsymbol{\ell}_n \in (\mathbb{R}_+)^T \text{ s.t } \sum_t \ell_{n,t} = E_n\}$ where $\ell_{n,t}$ represents the flow of n on arc t. The addition of time-dependent bounding constraints (6b), also considered in [10, 27], gives a more accurate model for electrical loads.

Another important practical example that fits in our context, considered in [4, 6] is given below.

Ex. 4. *Thermostatically controlled load :*

$$\mathcal{L}_n \stackrel{\Delta}{=} \{\underline{\theta}^{comf}_{n,t} \leqslant \theta_{n,t} \leqslant \overline{\theta}^{comf}_{n,t} \ , \forall t \in \mathcal{T}, \tag{7a}$$
$$\theta_{n,t} = \theta_{n,t-1} + \rho_n(\theta^{out}_{n,t} - \theta_{n,t-1}) + \varepsilon_n \ell_{n,t}, \forall t \in \mathcal{T}\}. \tag{7b}$$

Constraints (7) *offer a model for thermostatically controlled loads such as fridges or air conditioning. Here,* (7a) *ensures that the temperature remains within the comfort range* $[\underline{\theta}^{comf}_{n,t}, \underline{\theta}^{comf}_{n,t}]$. *The temperature evolves through the linear equation* (7b) *according to the efficiency parameters* ρ_n *and* ε_n*, and to the exterior temperature* $\theta^{out}_{n,t}$ *(see [6] for details). Using* (7b)*, one can rewrite* (7) *only with the variables* $(\boldsymbol{\ell}_{n,t})_{t\in\mathcal{T}}$.

Remark 2. *Owing to the convexity of* $\mathcal{L}_n$ *(Assumption 4), we do not consider appliances that require a fixed consumption profile but for which the starting time can be optimized (e.g. washing machines). In this case, one can use a (nonconvex) mixed-integer formulation, as in [28].*

We denote by $\mathcal{L} \stackrel{\Delta}{=} \mathcal{L}_1 \times \cdots \times \mathcal{L}_N$ the Cartesian product of the feasible sets. As b_n depends both on $\boldsymbol{\ell}_n$ and $\boldsymbol{\ell}_{-n}$, we get a N-person minimization game that we write in the standard form [29] as $\mathcal{G} \stackrel{\Delta}{=} (\mathcal{N}, \mathcal{L}, (b_n)_n)$.

E. Equilibrium Analysis and Efficiency

In game-theoretic models, a desirable stability property is when each player n has no interest to deviate unilaterally from her current profile $\boldsymbol{\ell}_n$. This corresponds to the notion of Nash Equilibrium (NE), that is:

Def. 1 (Nash, 1950)**. *Nash Equilibrium (NE).***
$(\boldsymbol{\ell}^{\text{NE}}_n)$ *is a* NE *of the minimization game* $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ *iff for any player* $n \in \mathcal{N}$*:*

$$\forall \boldsymbol{\ell}_n \in \mathcal{L}_n, \ b_n(\boldsymbol{\ell}^{\text{NE}}_n, \boldsymbol{\ell}^{\text{NE}}_{-n}) \leqslant b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}^{\text{NE}}_{-n}) \ .$$

[1] E_n can be set by the consumer, induced by the physical parameters of her appliances (battery capacity), or computed by learning the consumer's habits.

It is known that an NE may not exist or may not be unique, even in routing congestion games [15]. In our framework however, both properties are ensured, as stated below.

Theorem 1. *Under Assumption 1, there exists an NE of* $\mathcal{G}$.

Proof. This is a corollary of Rosen [17] as $\mathcal{G}$ is convex. □

To ensure the uniqueness of the NE, a common approach, adopted in [10], is to verify that a game is "diagonally strictly convex" [17]. We will see further from Rm. 7 and Prop. 1 that this property holds with Assumption 2. However, the uniqueness results based on this approach ask for more demanding conditions on the price functions $c_t(.)$ than Assumption 1. Here, in the case of feasibility sets of the form (6), Thm. 2 ensures the uniqueness for arbitrary convex and strictly increasing prices (Assumption 1).

Theorem 2. *Under Assumption 1 and if, for each* $n \in \mathcal{N}$, $\mathcal{L}_n$ *is of the form* (6)*, then* $\mathcal{G}$ *has a unique NE.*

Proof: See Appendix A. This proof extends the uniqueness theorem given in [15] in presence of the constraint (6b).

As said above, an NE is a very interesting situation in practice because of its stability: each player will only increase her bill if she changes her profile. However, an NE does not necessarily minimize the *social cost*

$$\text{SC}(\boldsymbol{\ell}) \stackrel{\Delta}{=} \sum_n b_n(\boldsymbol{\ell}) \ . \tag{8}$$

Remark 3. *With the billing equation* (4)*,* $\text{SC}(\boldsymbol{\ell})$ *is equal to the total* system cost $\sum_t L_t c_t(L_t)$*, a quantity that the aggregator should minimize. In general, the system cost can differ from the social cost of consumers, for instance if we consider that the aggregator makes a positive profit, or if we consider consumers utility functions as done in [23].*

In general games, an NE can be suboptimal in terms of SC. To measure the inefficiency of Nash Equilibria in terms of social cost, a standard quantity is the Price of Anarchy:

Def. 2 (Koutsoupias *et al*, 1999)**. *Price of Anarchy (PoA).***
Given a N-player game $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ *and* $\mathcal{L}_{\text{NE}}$ *its set of Nash equilibria, the PoA is defined as the following ratio:*

$$\text{PoA}(\mathcal{G}) = \frac{\sup_{\boldsymbol{\ell}\in\mathcal{L}_{\text{NE}}} \text{SC}(\boldsymbol{\ell})}{\inf_{\boldsymbol{\ell}\in\mathcal{L}} \text{SC}(\boldsymbol{\ell})} \ .$$

Note that, from Def. 2, as $\mathcal{L}_{\text{NE}} \subset \mathcal{L}$, the PoA is always greater than one. Furthermore, finding an upper bound on the PoA ensures that the social cost at any NE will be relatively close to the minimal social cost. Bounding the PoA is a hard theoretical question in general congestion games [32, 33]. In [34], the authors give an upper bound if the price functions are polynomial with bounded degree and positive coefficients. With degree one (affine prices, Assumption 3) the bound is $\text{PoA} \leqslant 1.5$, which means that the NE profile can induce costs as much as 50% higher than the optimal costs: implementing such a framework would not be worthwhile for the aggregator, as uncoordinated consumers will probably perform better (in our simulations, the uncoordinated profiles induce costs 16% higher than the optimal costs, see Tab. II). However, the results in [34] are worst-case bounds and these bounds are only approached asymptotically[2]: in our simulations with affine prices, the PoA was always much lower than 1.5 (around 1.017 from Tab. II). One of the reasons is that in [34] the model does not consider the power constraints (6b), and a PoA of 1.5 might be reached in our case only if the constraints (6b) are coarse enough. To further explain the low PoA in our instances, we found the following theorem by applying the (λ, μ) local smoothness technique of [34]:

Theorem 3. *Under Assumption 3, define for any $t \in \mathcal{T}$, $\varphi_t = (1 + \frac{\alpha_t}{\beta_t \overline{L}_t})^2$, where $\overline{L}_t = \sum_n \overline{\ell}_{n,t}$ and $t_0 \triangleq \arg\min_t \frac{\alpha_t}{\beta_t \overline{L}_t}$. Assuming that, for all $t \in \mathcal{T}$:*

$$\varphi_t \leqslant \varphi_{t_0} + 2 + \sqrt{1 + \varphi_{t_0}} \ , \tag{9}$$

the following inequality holds:

$$\mathrm{PoA}(\mathcal{G}) \leqslant \frac{1}{2}\left(1 + \sqrt{1 + \varphi_{t_0}^{-1}} + \frac{1}{2}\varphi_{t_0}^{-\frac{1}{2}}\right) \ . \tag{10}$$

Proof: See Appendix B.

Remark 4. *Using the inequality $\forall x \geqslant 0,\ \sqrt{1+x^2} \leqslant 1 + x$, (10) implies the following simplified—but coarser—bound:*

$$\mathrm{PoA}(\mathcal{G}) \leqslant 1 + \tfrac{3}{4} \sup_{t \in \mathcal{T}} \left(1 + \tfrac{\alpha_t}{\beta_t \overline{L}_t}\right)^{-1} \ . \tag{11}$$

The assumption (9) in Thm. 3 ensures that price functions (c_t) cannot differ too much from one time period to another. This is verified for instance if the price functions are uniform over $\mathcal{T}$ (*i.e.* $\forall t, c_t = c$). One can see that, according to Thm. 3, the PoA converges to one when $\alpha_t/(\beta_t \overline{L}_t)$ converges to infinity for each t: the PoA can be arbitrarily close to one if we choose the coefficients α_t large enough. This result is indeed intuitive: when the prices are constant ($\beta_t = 0$), they do not depend on the aggregates $\boldsymbol{L}$ and there is no congestion effect; the optimal profile is obtained by each consumer choosing the time periods with lowest prices, independently of $\boldsymbol{\ell}_{-n}$. Another interesting result is that the PoA also converges to one when the total load is low ($\forall t, \overline{L}_t \to 0$). Note that the right-hand-side of inequality (10) is decreasing with φ_0 and is equal to $(\frac{1+\sqrt{2}}{2})^2 \approx 1.457$ for $\varphi_0 = 1$ so our result is always tighter than the bound of 1.5 given in [34]. However, in our simulations with linear prices, the PoA was still lower than the bound (10), even when assumption (9) does not hold: the inequality (10) gives PoA $\leqslant$ 1.271 (average on the simulated days), while the PoA on mean values from Tab. II is 1.017. In this regards, getting a tighter bound or generalizing Thm. 3 to more general price functions could be the subject of future work.

III. Fast Computation of the Nash Equilibrium

Since we have shown that the NE is a good decentralized optimization target, the next question we address is the computation of the NE consumption profiles. This question is a central problem in game theory [35]. Furthermore, this computation has to be done in a short time to be implemented in practice. In this section, we provide two algorithms, we prove their convergence to the NE and we give a guarantee on their convergence rate in our specific setting.

A. Two Decentralized Algorithms

Given a profile $\boldsymbol{\ell}_{-n}$ of the others, consumer n chooses the profile $\boldsymbol{\ell}_n$ corresponding to a minimizer of (5), which is called her *Best Response*[3]. It is denoted by

$$\mathrm{BR}_n : \boldsymbol{s}_n \mapsto \operatorname*{argmin}_{\boldsymbol{\ell}_n \in \mathcal{L}_n} \sum_t \ell_{n,t} c_t(s_{n,t} + \ell_{n,t}) \ , \tag{12}$$

which only depends on the sum of the load of the others $\boldsymbol{s}_n \triangleq \sum_{m \neq n} \boldsymbol{\ell}_m \in \mathbb{R}^T$ because of the "aggregated" structure[4]. A natural algorithm to compute an NE is to iterate best responses and update the strategies, cycling over the set of users until convergence. This procedure, referred to as *Cycling Best-Response Dynamics* (CBRD) [36] is described by Algo 1.

Algo. 1 Cycling Best Response Dynamics (CBRD)

Require: $\boldsymbol{\ell}^{(0)}$, stopping criterion
1: $k \leftarrow 0$
2: **while** stopping criterion not true **do**
3: **for** $n = 1$ to N **do**
4: $\boldsymbol{s}_n^{(k)} = \sum_{m<n} \boldsymbol{\ell}_m^{(k+1)} + \sum_{m>n} \boldsymbol{\ell}_m^{(k)}$
5: $\boldsymbol{\ell}_n^{(k+1)} \leftarrow \mathrm{BR}_n(\boldsymbol{s}_n^{(k)})$
6: **end for**
7: $k \leftarrow k + 1$
8: **end while**

Standard stopping criteria that can be used in Algo 1 are a maximum number of iterations $k_{\max}$, a maximum CPU time, an objective on the distance between iterates $\left\|\boldsymbol{\ell}^{(k-1)} - \boldsymbol{\ell}^{(k)}\right\| \leqslant \varepsilon_{\text{stop}}$, or the satisfaction of the KKT conditions of optimality for each user's convex optimization problem (5) up to an absolute error tolerance.

In Algo 1, the only computationally demanding step is the computation of $\mathrm{BR}_n(\boldsymbol{s}_n)$ on Line 5. Its complexity differs according to the price functions c_t and the feasibility set $\mathcal{L}_n$. In general, there is no explicit expression of $\mathrm{BR}_n(\boldsymbol{s}_n)$ but, as $\mathcal{L}_n$ is convex and $\boldsymbol{\ell}_n \mapsto b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n})$ is convex, techniques of nonlinear convex optimization can be used to find an approximating solution [26]. The problem simplifies if prices are affine (Assumption 3) and $\mathcal{L}_n$ is given by (6a)-(6b) and none of the bounding constraints (6b) is active. In that case, an explicit expression of $\mathrm{BR}_n(\boldsymbol{s}_n)$ can be found [37] so Line 5 can be executed in constant time. In the general case of feasibility sets of the form (6) (bounding constraints (6b) can be active), we are still in a specific case of quadratic programming where an exact solution can be computed in $\mathcal{O}(T)$ with [38]. When $\mathcal{L}_n$ is a general polytope given as a set of linear inequalities (as in [4]), convex quadratic programming [26] can be used to compute the solution.

Remark 5. *The **for** loop in Algo 1 (Line 3) implements sequential updates and cycles over the set of players in the arbitrary order $1, 2, \dots, N$ in a Gauss-Seidel manner [3].*

[2]Meaning that there exists a sequence of games $(\mathcal{G}_\nu)_{\nu \geqslant 0}$, with parameters depending on ν, and affine price functions c_t such that $\mathrm{PoA}(\mathcal{G}_\nu) \underset{\nu \to \infty}{\longrightarrow} 1.5$.

[3]As player n chooses her best profile to the fixed profiles of the others; she responds to them.

[4]In a general setting, BR_n would be a function of $\boldsymbol{\ell}_{-n}$, and can be multivalued. In that case, we can still use Algo 1 by arbitrarily choosing any element of $\mathrm{BR}_n(\boldsymbol{s}_n)$ at Line 5.

Choosing a "good" order of the BR *in the* **for** *loop might accelerate the convergence of the algorithm. A* simultaneous *version of Algo 1 (without Line 4 and with Line 5 executed by all players in parallel) could also improve the speed of Algo 1, but we observed that doing so can prevent its convergence.*

Another natural algorithm to compute the equilibrium is to emulate the projected gradient descent, well-known in convex optimization [39], by considering the gradient of each objective function of the players, as described in Algo 2.

Algo. 2 Simultaneous Improving Response Dynamics (SIRD)

Require: $\boldsymbol{\ell}^{(0)}$, γ, stopping criterion

1: $k \leftarrow 0$
2: **while** stopping criterion not true **do**
3: **for** $n = 1$ to N **do**
4: $\boldsymbol{\ell}_n^{(k+1)} \leftarrow \Pi_{\mathcal{L}_n}\left(\boldsymbol{\ell}_n^{(k)} - \gamma \nabla_n b_n(\boldsymbol{\ell}_n^{(k)}, \boldsymbol{\ell}_{-n}^{(k)})\right)$
5: **end for**
6: $k \leftarrow k+1$
7: **end while**

At Line 4 of Algo 2, $\Pi_{\mathcal{L}_n}$ denotes the projection on the feasibility set $\mathcal{L}_n$ of consumer n and $\nabla_n b_n = (\partial b_n / \partial \ell_{n,t})_{t\in\mathcal{T}}$ denotes the gradient with respect to variable $\boldsymbol{\ell}_n$. The same stopping criteria listed below Algo 1 can be used for Algo 2. The chosen denomination *improving response* recalls that, at each iteration of Algo 2, player n *improves* her profile $\boldsymbol{\ell}_n$ by performing a projected gradient step (Line 4), but in general does not choose the best improvement as in Algo 1.

Note that from Algo 1 to Algo 2, only the instructions within the **for** loop are changed: the update of $\boldsymbol{s}_n$ and computation of BR_n (Lines 4 and 5 of Algo 1) are replaced with the gradient step (Line 4 of Algo 2).

Remark 6. *Both Algo 1 and Algo 2 can be implemented in a "decentralized" procedure: Lines 4 and 5 in Algo 1 and Line 4 in Algo 2 can be performed locally by each consumer's ECS. In this way, consumers' privacy is respected as they do not send any information about their constraints to the aggregator. On the other hand, they only receive information on the aggregated load* $\boldsymbol{s}_n^{(k)}$ *and can hardly deduce the individual consumption profiles* $\boldsymbol{\ell}_{-n}$ *of the other consumers.*

The computational complexity of one iteration (within the **for** loop) of Algo 2 is equivalent to the complexity of the projection $\Pi_{\mathcal{L}_n}$, which can be computed with the Quadratic Program (QP) $\Pi_{\mathcal{L}_n}(\boldsymbol{\ell}'_n) = \operatorname{argmin}_{\boldsymbol{\ell}_n\in\mathcal{L}_n} \|\boldsymbol{\ell}'_n - \boldsymbol{\ell}_n\|_2^2$ so it would be of the same order of complexity (see [26, Lecture 4]) as one iteration (within the **for** loop) of algorithm CBRD with affine prices (Assumption 3). With the specific structure (6) of the feasible set $\mathcal{L}_n \subset \mathbb{R}^T$, a QP can be solved very efficiently in $\mathcal{O}(T)$ [38], so that one iteration of Algo 2 will be very fast. Moreover, as we do not update sequentially the load of the others $\boldsymbol{\ell}_{-n}$ in Algo 2, the projected gradient step within the **for** loop can be computed simultaneously and can be parallelized.

B. Game Stability and Convergence of Algos 1 and 2

In this section, we provide theoretical convergence rates of the two algorithms proposed in Sec. III-A. We first recall the notion of *stability*, and prove (Prop. 1) that the energy consumption game $\mathcal{G}$ defined above is *strongly stable* under Assumption 2. The notion of stability was introduced in [40] in order to study different game dynamics in continuous time and their convergence to NE. We extend this property to a "strong" version (symmetrically to the concept of strong monotonicity for operators).

Def. 3 (Hofbauer and Sandholm, 2009)**. *Stable Game.*** *A minimization game* $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ *is stable iff*

$$\forall \boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L},\ (\boldsymbol{\ell}' - \boldsymbol{\ell})^T.\,(F(\boldsymbol{\ell}') - F(\boldsymbol{\ell})) \geqslant 0\ , \qquad (13)$$

with $F(\boldsymbol{\ell}) \triangleq (\nabla_n b_n(\boldsymbol{\ell}))_{n\in\mathcal{N}}$.
Moreover, $\mathcal{G}$ *is* a*-strongly stable, with a constant* $a > 0$*, iff:*

$$\forall \boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L},\ (\boldsymbol{\ell}' - \boldsymbol{\ell})^T.\,(F(\boldsymbol{\ell}') - F(\boldsymbol{\ell})) \geqslant a\,\|\boldsymbol{\ell} - \boldsymbol{\ell}'\|^2\ . \qquad (14)$$

Remark 7. *The condition of stability in* (13) *is equivalent to the condition of strict diagonal convexity in [17], which implies uniqueness of* NE *[17, Thm.2].*

Def. 3 gives an abstract condition on an operator that depends on the objective functions of the players. In our case, players objectives (b_n) depend linearly on price functions $(c_t)_t$ through (4), so it is interesting to translate the condition of Def. 3 directly on the price functions, as stated in Prop. 1.

Proposition 1. *Let* $a > 0$ *such that Assumption 2 holds. Then, the game* $\mathcal{G}$ *is* a*-strongly stable.*

Proof: See Appendix Sec. C.

This property will be used to show the convergence of Algo 2 in Thm. 5. Concerning Algo 1, the approach is different and the convergence is established only in the specific case of Assumption 3. In general games, CBRD might not converge [41] or might take an exponential time to converge [42]. In atomic splittable congestion games on a parallel network, as in our case, the convergence and the speed of Algo 1 has been studied previously in [19] and [20], where the authors show by different methods that there is a geometric convergence in the case of $N = 2$ players and convex and strictly increasing price functions (Assumption 1). However, to the best of our knowledge, the convergence in this setting and for more players $N > 2$ is still an open question.

In our case, simulations show a geometric convergence rate for any instance of $\mathcal{G}$ satisfying Assumption 3 and for any $N \in \mathbb{N}$, as illustrated in Fig. 1. In [20], it is conjectured that this geometric convergence may also hold under Assumption 1. Restricting ourselves to affine price functions, we notice that game $\mathcal{G}$ is a potential game [23, 43] and we get the following guarantee on the rate of convergence of Algo 1:

Theorem 4. *Under Assumption 3, the sequence of iterates of Algorithm CBRD* $\left(\boldsymbol{\ell}^{(k)}\right)_{k\geqslant 0}$ *converges to the unique* NE $\boldsymbol{\ell}^{\mathrm{NE}}$ *of* $\mathcal{G}$*. Moreover, the convergence rate satisfies:*

$$\forall k \geqslant 0,\ \left\|\boldsymbol{\ell}^{\mathrm{NE}} - \boldsymbol{\ell}^{(k)}\right\|_2 \leqslant C\sqrt{\frac{M}{a}} \times \frac{N}{\sqrt{k}}\ , \qquad (15)$$

where C depends on $\ell^{(0)}$ and the billing functions,
$M = 2\max_t \beta_t$ *and* $a = 2\min_t \beta_t$.

Proof: See Appendix D. The result is implied by convergence of alternating block coordinate minimization method [44].

The proof of Thm. 4 uses the fact that $M = \max_n M_n$ where M_n is a Lipschitz constant of $\nabla_n b_n$, and a is a strong convexity (and a-strong stability) constant. To the best of our knowledge, the question to know if Thm. 4 holds for general price functions is open; it can be an avenue for future research.

It is easier to get a strong guarantee on the convergence rate of Algo 2 for general price functions, as stated in Thm. 5:

Theorem 5. *Denote by M_n a Lipschitz constant of $\nabla_n b_n$ and $M \stackrel{\Delta}{=} \max_n M_n$. Under Assumption 2 (a- strong stability), for a step $\gamma \stackrel{\Delta}{=} a/(NM^2)$, SIRD converges to the NE. Moreover:*

$$\forall k \geqslant 0, \left\|\boldsymbol{\ell}^{\mathrm{NE}} - \boldsymbol{\ell}^{(k)}\right\|_2 \leqslant \left(1 - \tfrac{a^2}{NM^2}\right)^k \left\|\boldsymbol{\ell}^{\mathrm{NE}} - \boldsymbol{\ell}^{(0)}\right\|_2 . \quad (16)$$

Proof: See Appendix Sec. E.

Note that, under Assumption 3, as stated in Thm. 5 we have $M = 2\max_t \beta_t$ and $a = 2\min_t \beta_t$, which gives the explicit contraction ratio $\eta = 1 - \frac{\max_t \beta_t}{N \min_t \beta_t}$.

The bound given in Thm. 5 shows that the convergence of Algo 2 is slower when the number of consumers N increases.

C. Numerical Convergence and Comparisons

In this section, we present a numerical comparison of the two algorithms CBRD and SIRD given above. We also add two other algorithms from related papers in the comparison:

- the distributed iterative proximal-point algorithm [10, Algo. 1], referred to as *itProxPt*. This algorithm is analogous to the SIRD algorithm proposed here, but with a decreasing time-step (as opposed to the constant step γ) and a regularization proximal term. Line 4 of Algo 2 is replaced with:

$$\boldsymbol{\ell}_n^{(k+1)} \leftarrow \Pi_{\mathcal{L}_n}\Big[\boldsymbol{\ell}_n^{(k)} - \gamma_k \left(\nabla_n b_n(\boldsymbol{\ell}_n^{(k)}, \boldsymbol{\ell}_{-n}^{(k)}) + \theta(\boldsymbol{\ell}_n^{(k)} - \boldsymbol{\ell}_n^{(k-1)})\right)\Big].$$

In the numerical results below, we choose $\gamma^{(k)} = k^{-0.52}$ (to ensure the convergence criterion of $\sum \gamma_k < \infty$ and $\sum \gamma_k^2 = +\infty$ while keeping sufficiently large steps) and a regularization weight $\theta = 0.5$ (according to our tests, this latter parameter does not have a significant impact on the speed of convergence);

- the proximal decomposition algorithm [11, Algo. 1] referred to as *proxBR*. This algorithm is analogous to the CBRD (Algo 1) proposed in this paper, with a proximal regularization term and an additional loop to update this proximal term. Namely, the authors introduced a "regularized" game where each player's objective is replaced by:

$$f_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) = b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) + \tfrac{\tau}{2}\left\|\boldsymbol{\ell}_n - \bar{\boldsymbol{\ell}}_n\right\|_2^2 , \quad (17)$$

where $\bar{\boldsymbol{\ell}}_n$ is the "centroid" updated in an additional loop. The idea of the algorithm is to compute the NE of this regularized game, and update the centroid to the computed NE. Of course, the NE of the regularized game can only be computed approximately. In the numerical results below, we choose to update the centroid when the distance between the iterates of two BR cycles $\left\|\boldsymbol{\ell}^{(k+1)} - \boldsymbol{\ell}^{(k)}\right\|_2$ is lower than 10^{-4}. The regularization parameter is taken to $\tau = 3(N-1)\max_t c_t$, just high enough to ensure the given condition of convergence (choosing a higher parameter τ slows the convergence). Note that CBRD corresponds to the case $\tau = 0$, a case dismissed by the proposed convergence conditions [11, Thm. 2].

We consider numerical instances with $T = 10$, and feasibility sets of the form (6), constructed as follows:

1) in instances **I1**, functions (c_t) are affine and uniform *i.e.* $\forall t, c_t(L) = c(L) = \alpha + \beta L$ where α (resp. β) is drawn uniformly from $[0, 4]$ (resp. $[1, 4]$). For each $n \in \mathcal{N}$, E_n is drawn uniformly from $[1, 10]$. The lower bounds are all set to $\underline{\ell}_{n,t} = 0$. A subset $\mathcal{T}_n$ of consecutive time periods of length $T_n \geqslant 4$ is drawn randomly from $\mathcal{T}$, and the upper bounds are set to $\overline{\ell}_{n,t} = E_n$ if $t \in \mathcal{T}_n$ and to $\overline{\ell}_{n,t} = 0$ if $t \notin \mathcal{T}_n$;
2) in instances **I2**, price functions are affine and time dependent *i.e.* $\forall t, c_t(L) = \alpha_t + \beta_t L$ where α_t (resp. β_t) is drawn uniformly from $[0, 4]$ (resp. $[1, 4]$). For each $n \in \mathcal{N}$, E_n is drawn uniformly from $[1, 10]$. A subset $\mathcal{T}_n$ of consecutive time periods of length $T_n \geqslant 4$ is drawn randomly from $\mathcal{T}$. For $t \notin \mathcal{T}_n$, we take $\underline{\ell}_{n,t} = \overline{\ell}_{n,t} = 0$. For $t \in \mathcal{T}_n$, $\underline{\ell}_{n,t}$ is drawn uniformly from $[0, E_n/T_n]$, and $\overline{\ell}_{n,t}$ is drawn uniformly from $[E_n/T_n, E_n]$, ensuring that $\mathcal{L}_n \neq \emptyset$.

The four algorithms considered above are implemented in Python 3.5 and run on an Intel i7 @2.6GHz on a single core and with 8GB of RAM. To solve the quadratic programs (QP) subproblems of each algorithm (Line 5 of CBRD and Line 4 of SIRD), we use the Brucker algorithm [38]. Note that if $\mathcal{L}_n$ is a general polytope, any quadratic programming solver can be used instead to solve those subproblems.

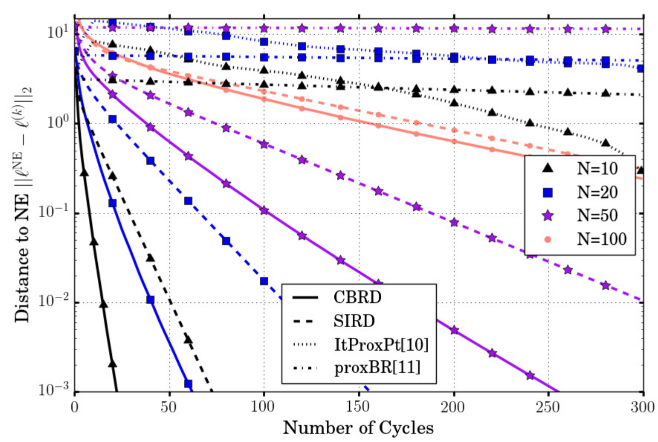

Fig. 1: Average convergence rate of the four implemented algorithms on ten instances **I1**. *When the number of players N increases, the convergence rate of both algorithms decreases, but SIRD becomes faster than CBRD.*

Fig. 1 shows the convergence of the four algorithms to the NE. The results are given on average on a set of ten instances **I1**. The convergence speed of the algorithms decreases with the number of users, as given in Thm. 5. We observe that, in spite of the weaker theoretical result for CBRD (see (15) compared to (16)), the convergence seems also geometric, and even faster than Algorithm SIRD. We also observe that the convergence rate of the comparison algorithms *itProxPt* and *proxBR* is not comparable with the convergence rate of

CBRD and SIRD. This can be explained by the addition of the regularization term which slows the convergence and, also, by the diminishing step for ItProxPoint (instead of a constant step γ for SIRD) and by the additional loop for proxBR compared to CBRD.

N=	10	20	50	100
BR	0.15	1.4	23.98	112.1 TL(97)
SIRD	0.47	4.11	41.97	TL(100)
ItProxPt[10]	2.07	17.18	89.3 TL(83)	TL(100)
proxBR[11]	57.68 TL(2)	98.4 TL(94)	TL(100)	TL(100)

(a) **I1** Uniform affine prices, bounds $\underline{\ell}_{n,t} = 0,\ \overline{\ell}_{n,t} \in \{0, E_n\}$

N=	10	20	50	100
BR	0.08	0.36	3.45	54.17 TL(3)
SIRD	2.24	12.08	95.5 TL(86)	TL(100)
ItProxPt[10]	0.77	4.83	65.1 TL(30)	TL(100)
proxBR[11]	75.28 TL(4)	TL(100)	TL(100)	TL(100)

(b) **I2** Random affine prices, random bounds $\underline{\ell}_{n,t}, \overline{\ell}_{n,t}$

TABLE I: Average CPU time (sec.) for NE computation for one hundred instances **I1** (a) and one hundred instances **I2** (b). *xx-TL(k) indicates that the time limit (120 sec.) was reached for k instances, while the remaining instances took an average CPU time of xx seconds.*

Tab. I shows the CPU time needed to compute the NE at a given precision: the stopping criterion considered here is the satisfaction of the KKT conditions for each problem (5) with an absolute error of 10^{-2}, with a time limit (TL) of two minutes per instance.

We observe that CBRD is the fastest method to compute the NE. The algorithm *ProxBR* reaches the time limit even for a relatively small number of consumers ($N = 20$). We notice that SIRD is slower on the heterogeneous instances **I2**, which can be easily explained from the step γ chosen in Thm. 5. On the contrary, CBRD and *itProxPt* are slower on homogeneous instances **I1**, which can be explained by the importance of symmetries in those instances. We can see that for the bigger instances ($N = 100$), the time limit is reached for half of the instances for CBRD (and for all instances for other algorithms). The time limit considered here was only of two minutes: it could be extended in a practical implementation of a DR program. However, the computational time can be limiting, for instance if the equilibrium needs to be recomputed in case of a change of parameters (see Sec. IV). Thus, using those methods for a larger system (thousands of consumers) might be prohibited, in particular if we allow a more complex description of users constraints than (6).

However, if we stay within the proposed order of magnitude ($N \leqslant 100$), the simulations show that CBRD (and SIRD in most cases) needs only a few seconds to compute the NE. It enables to consider an online procedure, where the equilibrium can be recomputed at each hour, as explained below.

IV. Simulation of Online Demand Response

In this section, we propose a practical procedure to implement the DR framework described above. We assume that, as described in Sec. II-C Example 3, the aggregator prices come from a generic cost function $C(.)$ that depends on the total load $L_{\text{NF},t} + L_t$ (nonflexible plus flexible) at each time period. The flexible consumption is considered as an additional load and its price is set for each time period $t \in \mathcal{T}$ to:

$$c_t(L_t) \triangleq \frac{1}{L_t} \left(C(L_{\text{NF},t} + L_t) - C(L_{\text{NF},t}) \right) \ . \tag{18}$$

The equilibrium consumption for the flexible consumption profiles have to be computed before real-time consumption. As a result, the nonflexible demand $\boldsymbol{L}_{\text{NF}}$ has to be estimated in order to evaluate price functions $(c_t)_t$ by injecting the estimation $\hat{\boldsymbol{L}}_{\text{NF}}$ in (18). To minimize the impact of forecast errors made on $\boldsymbol{L}_{\text{NF}}$, we consider an online procedure in which, at each hour, an updated forecast $\hat{\boldsymbol{L}}_{\text{NF}}$ is taken into account. The equilibrium profiles for the flexible consumption is then re-computed for the hours ahead to the end of the optimization horizon T, using Algo 1 or Algo 2.

A. Online Demand Response Procedure

The initial time horizon $\mathcal{T}$ that we consider for the planning via DR starts each day at noon ($t = 1$) and stops at noon the day after ($t = T$), with an hourly time step. The "online" procedure computes the DR equilibrium flexible consumption profiles on time horizon $\{1, \dots T\}$ for each day. As the price functions c_t depend on the nonflexible load through (18), and as the accuracy of forecast of this load improves when approaching from real-time, we re-compute the equilibrium using updated forecasts at each time period, as described below.

Algo. 3 Online Demand Response Procedure

1: Start at $t = 1$
2: **while** $t \leqslant T$ **do**
3: Set new horizon $\mathcal{T}^{(t)} = \{t, t+1, \dots, T\}$
4: Get $\boldsymbol{L}_{\text{NF}}$ forecast on $\mathcal{T}^{(t)}$: $\hat{\boldsymbol{L}}^{(t)}_{\text{NF}} \triangleq \big(\hat{L}^{(t)}_{\text{NF},s}\big)_{t \leqslant s \leqslant T}$
5: Re-compute prices $c_t(.)$ for $t \in \mathcal{T}^{(t)}$ from (18)
6: Run Algo. SIRD or BRD to compute NE $\boldsymbol{\ell}^{(t)}$ on $\mathcal{T}^{(t)}$
7: **for** each user $n \in \mathcal{N}$ **do**
8: Realize computed profile on time t, $\ell^{(t)}_{n,t}$
9: Update $\mathcal{L}^{(t+1)}_n \triangleq \big\{ (\ell_{n,s})_{s>t} | (\ell^{(t)}_{n,t}, [\ell_{n,s}]_{s>t}) \in \mathcal{L}^{(t)}_n \big\}$
10: **end for**
11: Wait for $t + 1$
12: **end while**

Remark 8. *If one considers sets $(\mathcal{L}_n)_n$ of the form* (6)*, then the updating step on Line 9 only consists in updating the energy demand for the remaining time: $E^{(t+1)}_n \triangleq E^{(t)}_n - \ell^{(t)}_{n,t}$.*

Remark 9. *In practice, the NE profile $\boldsymbol{\ell}^{(t)}$ has to be computed before period t to begin consumption at time t (Line 8). If τ is an upper bound on the computation time of the NE profile (Line 6), then, as we want to use the latest available forecast, Lines 3-5 would be run just before $t - \tau$, Line 6 is run in the interval $[t - \tau, t]$ and Line 8 is executed through $[t, t + 1]$.*

Observe that in Algo 3, considering an updated forecast at Line 4 leads to updated price functions $(c_t)_t$ (Line 5), according to equation (18). In turn, the updated price functions modify the objective function of user n, b_n, used in Line 6.

The difference of Algo 3 with an "offline" version is that we recompute the equilibrium consumption (Line 6) at each time for all the time periods ahead. In an offline DR, we would compute the equilibrium consumption for all the horizon $\mathcal{T} = \{1, \dots, T\}$ *only once*, just before $t = 1$.

Proceeding with this "online" version has two main advantages. First, it enables to rely on updated forecasts with new information acquired on the nonflexible load $\boldsymbol{L}_{\text{NF}}$ (Line 4). Second, it also enables to cope with local issues as disconnection of an user or a communication bug: in that case, lines 8 and 9 would not be executed for the involved user, and this user will have the same energy demand for the next round at $t+1$. With this kind of online procedure, it is also important to ensure that the final realized profile $(\ell_{n,t}^{(t)})_{t\in\mathcal{T}}$ is globally consistent: in the limit of perfect forecasts, it has to correspond to an equilibrium of the initial game.

Theorem 6. *Suppose that either Assumption 2 with $a = 0$ holds (strict stability from Prop. 1) or that Assumption 1 holds and the sets $(\mathcal{L}_n)_n$ are of the form* (6)*. Then the online DR procedure of Algo 3 is consistent: if forecasts are perfect (i.e. $\forall t \in \mathcal{T}, \forall t' \in \mathcal{T}^{(t)}, \hat{L}_{\text{NF},t'}^{(t)} = L_{\text{NF},t'}$), then for any $t_2 > t_1$, the NE profile $\boldsymbol{\ell}^{(t_1)}$ computed at t_1 with forecast $\hat{\boldsymbol{L}}_{\text{NF}}^{(t_1)}$ is equal on $\{t_2, \dots, T\}$ to the NE profile $\boldsymbol{\ell}^{(t_2)}$ computed at t_2 with forecast $\hat{\boldsymbol{L}}_{\text{NF}}^{(t_2)}$.*

Proof: See Appendix F.

Thm. 6 states a *dynamic programming principle* adapted to our game-theoretic framework. To quantify the value of this online procedure in the more realistic case of imperfect forecasts, we simulate it on a set of consumers and parameters taken from real data, defined below.

B. Consumers

We consider a set of $N = 30$ users owning an Electric Vehicle (EV) from the database of Texan residential consumers *PecanStreet Inc.* [45]. We consider that the charging of the EV is the only flexible appliance of each consumer managed through the DR program, while the remaining of the user's consumption is nonflexible and is taken as in the data. We denote by $\mathcal{D} \triangleq \{16/01/01, \dots, 16/01/31\}$ the set of the 31 days of January 2016 for which we simulate the DR program and we index a parameter by $d \in \mathcal{D}$ when it is specific to day d. For constraints (6a-6b), we take, for each day $d \in \mathcal{D}$, the total flexible demand of user n, $E_{n,d}$ as the total observed consumption for the EV of n on the time set $\mathcal{T} = \{1, \dots, T\}$, taken as the twenty-four hours from day d 12PM to day $d+1$ 11AM (including the regular EV residential night charging period). The power lower bound is always taken to zero $\underline{\ell}_{n,d,t} = 0$. For the power upper bound $\overline{\ell}_{n,d,t}$, we consider two cases: if a positive power was given at d, t in the data, we set $\overline{\ell}_{n,d,t}$ to the maximum power given to n's EV on all time periods in the data in the set $\mathcal{D}$. If the power given to the EV is 0 at d, t in the data, we take $\overline{\ell}_{n,d,t} = 0$ (*i.e.* we consider that the EV of n was not available to charge on period d, t).

C. Price Functions and Forecasts of the Nonflexible Load

Following [23], we consider that the aggregator has a providing cost for the global demand at time t, $D_t \triangleq (L_{\text{NF},t} + L_t)$, that does not depend on the time, and given (in \$) by $C(D_t) \triangleq 0.711 - 0.0417 D_t + 0.00295 D_t^2$ where the coefficients replicate the cost function of a real residential electricity provider. For this, we computed the average, minimum and maximum values of $L_{\text{NF},t}$ over all the hours of the 31 days of January 2016 on our set of 30 consumers and interpolate the three values (avg $L_{\text{NF},t}$, $\min L_{\text{NF},t}$, $\max L_{\text{NF},t}$) to three respective prices proposed by the Texan distributor *Coserv* [46] so that the per-unit price $\tilde{c}(D) \triangleq C(D)/D$ verifies $\tilde{c}(\text{avg}\, L_{\text{NF},t}) = 0.080\$/\text{kWh}$ (price for "base" contracts), $\tilde{c}(\min L_{\text{NF},t}) = 0.055\$/\text{kWh}$ (price for Off-Peak hours in Time-of-Use contracts) and $\tilde{c}(\max L_{\text{NF},t}) = 0.14\$/\text{kWh}$ (price for Peak hours). Following (18), the price for the flexible load is given by: $c_t(L_t) = (-4.17 + 0.590 L_{\text{NF},t}) + 0.295 L_t$, so that Assumption 3 holds.

As prices depend on the nonflexible load, the aggregator has to compute a forecast $\hat{\boldsymbol{L}}_{\text{NF}}^{(t)} \triangleq (\hat{L}_{\text{NF},t}^{(t)}, \dots \hat{L}_{\text{NF},T}^{(t)})$ at each time t so that the equilibrium consumption for time periods $\{t, \dots, T\}$ can be computed using Algo 1 or Algo 2. To simulate the forecasts, we assume that the forecast made at time t for period $t' \geqslant t$, $\hat{L}_{\text{NF},t'}^{(t)}$ has no bias, that is $\mathbb{E}[L_{\text{NF},t'} | \sigma(\mathcal{F}_t)] = \hat{L}_{\text{NF},t'}^{(t)}$ (where $\mathcal{F}_t$ is the natural filtration over $(L_{\text{NF},t})_t$), and that we have perfect information at time t, that is: $\hat{L}_{\text{NF},t}^{(t)} = L_{\text{NF},t}$. Considering that $L_{\text{NF},t} = P_t e^{X_t}$ where X_t follows an Ornstein-Uhlenbeck [47] process with mean reverting coefficient m and volatility σ, and P_t a seasonality factor that depends on the hour of the week (1st hour to 168th hour), we get for any $t \leqslant t'$:

$$\hat{L}_{\text{NF},t'}^{(t)} = P_{t'} \left(\frac{L_{\text{NF},t}}{P_t} \right)^{e^{-m(t'-t)}} \exp\left(\frac{\sigma^2}{4m} (1 - e^{-2m(t'-t)}) \right).$$

Using a least-squares regression on the observed data from years 2014 and 2015, we compute $m \simeq 0.198\ \text{h}^{-1}$ and $\sigma \simeq 0.117\ \text{h}^{-1/2}$. An example of the simulated forecasts made at four different time periods is given in Fig. 2.

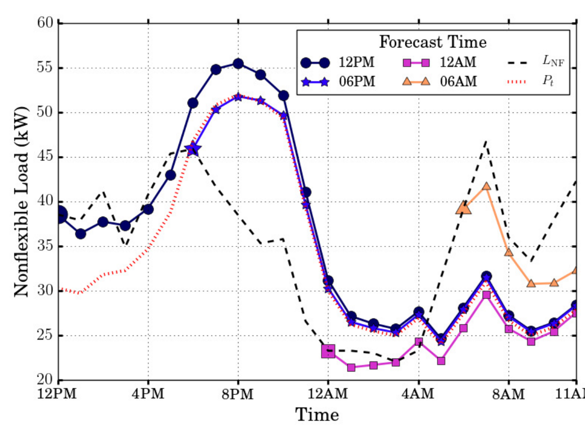

Fig. 2: Forecasts of the nonflexible load $\hat{\boldsymbol{L}}_{\text{NF}}^{(t)}$ evolving in time. *We assume a perfect forecast at time t for t. Forecasting performance increases when approaching real time.*

D. Gains with the Online DR Procedure

For each day of January 2016, we run the online DR Procedure described in Sec. IV-A to get the flexible consumption profile of each user $\boldsymbol{\ell}_n$, and the associated social cost on the DR horizon $\{1, \dots, T\}$. We compare the total social cost over the set $\mathcal{D}$ of simulated days obtained via the DR online procedure with the total social costs obtained with the four other consumption scenarios below:

1) *uncoordinated* case: no DR is implemented to control or incentivize consumers flexibility; the flexible consumption profiles are taken as the observed values in the data;

2) *offline DR*: the equilibrium is computed only once at $t = 1$ and for the whole time horizon $\{1, \dots T\}$ considering the first forecast $\hat{\boldsymbol{L}}^{(1)}_{\text{NF}}$ available at $t = 1$;

3) *perfect forecast DR*: offline DR, where we take $\hat{\boldsymbol{L}}^{(1)}_{\text{NF}} = \boldsymbol{L}_{\text{NF}}$. With Thm. 6, it is useless to recompute the profiles at each time period;

4) *optimal* scenario: a centralized entity (with perfect forecasts) computes the flexible consumption profile $\boldsymbol{\ell}$ that minimizes the system cost $\sum_t L_t c_t(L_t)$ (also equal to the social cost, Rm. 3).

For the online DR and the comparison scenarios 2) and 3), NE are computed by implementing Algo 1 (CBRD) with the same configuration than in Sec. III-C.For the comparison scenario 4), we compute the *optimal* consumption profile satisfying all users constraints (6a-6b), for each simulated day (from 12PM to 11AM) in $\mathcal{D}$. The associated problems are convex QPs with linear constraints, that are solved easily with the solver Cplex 12.6 in 0.31seconds on average.

Cons. Scenario	Social Cost	Avg. Price	Gain
Uncoordinated	\$ 1257.2	0.200 \$/kWh	—
Offline DR	\$ 1231.6	0.195 \$/kWh	2.036%
Online DR	**\$ 1131.1**	0.180 \$/kWh	**10.03%**
Perfect forecast DR	\$ 1075.2	0.171 \$/kWh	14.47%
Optimal scenario	\$ 1056.8	0.169 \$/kWh	15.94%

TABLE II: Social Costs, average prices and relative gain to the uncoordinated consumption scenario on January 2016 .

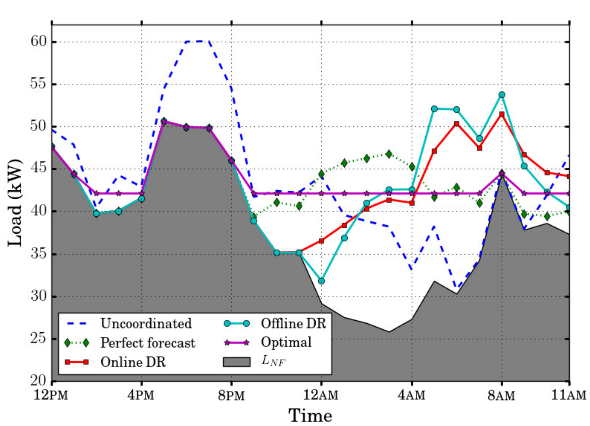

Fig. 3: Consumption profiles on a typical day, with the different scenarios listed in Sec. IV-D. *The optimal profile flattens the consumption. The online DR procedure of Algo 3 gets closer to the Perfect forecast (offline) DR profile.*

Tab. II summarizes the numerical results: it gives the total costs on the 31 days of January 2016 and compares the gains of the different flexible consumption scenarios relatively to the *uncoordinated* one. We first observe that performances of perfect forecasts DR are closed to the optimal scenario. This confirms the theoretical results provided on the efficiency of the NE in Thm. 3. We see on this table that the online DR procedure achieves significant savings compared to the offline version for which the performance is really low on average: using the offline DR decreases the system costs by 2% relatively to the uncoordinated profile, that is, when consumers behave without any incentives (comparison scenario 1)). Implementing this offline DR program might not be worthy as it still involves a sophisticated communication and automation structure and it adds more constraints for consumers. This low performance is directly linked to our simple and naive model for the nonflexible load forecasts, which results in inaccurate forecasts for the last hours, as seen in Fig. 2. Even if more advanced forecasting methods (see [48]) could improve this accuracy, we cannot get rid of the high variance due to the small number of consumers ($N = 30$ in our example, and several hundreds for an aggregator at the scale of a typical low-voltage station). The online DR procedure seems to bring a solution to this issue: even with our simple forecast model, we achieve more than 10% of savings, with a gap of only 4% from the scenario with perfect forecasts. These results show that implementing the given online DR procedure, even without very accurate forecasts, is worthwhile for the aggregator.

V. Conclusion

In this paper, we developed a game-theoretic model for a residential demand response program, and we addressed several issues both on the theoretical and practical aspects. We gave several new theoretical results about the uniqueness and existence of a Nash equilibrium consumption profile for which the price of anarchy is theoretically bounded. We proved that the two proposed algorithms CBRD and SIRD provide approximations of the NE at an arbitrary accuracy in finite time. We introduced and simulated an online procedure that recomputes the NE profiles at each time period to take into account new information, for example updated forecasts. We showed numerically that this online procedure achieves a small price of anarchy when the parameters are fixed but also when the demand is uncertain. Our simulations show that the online procedure reduces the impact of inaccurate forecasts on the social cost by 8%.

Several extensions of this work can be undertaken. First, our online procedure can be directly applied in the presence of other sources of stochasticity such as market prices or local renewable production sources. The aggregator objective can also be generalized to take into account the distance to a reference load profile or to maximize consumption during renewable production peaks or to take into account market prices. Also, two main theoretical questions are still open. First, the result on the PoA bound could be improved to be tighter to the numerical results, and generalized to a larger set of functions. Second, the convergence theorem for the Best Response Dynamics (CBRD) could also be improved, as the observed convergence rate is faster than the given bound, and the convergence is numerically observed for a larger set of prices than affine functions.

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Appendix A
Proof of Thm. 2: Uniqueness of NE in $\mathcal{G}$

The proof follows the one of [15], extending it to the constrained case with constraints of the form (6b).

We denote by $\lambda_n \in \mathbb{R}$ the Lagrange multiplier associated to (6a), along with $\underline{\mu}_{n,t} \geqslant 0$ (resp. $\overline{\mu}_{n,t} \geqslant 0$) the multiplier associated to $\underline{\ell}_{n,t} \leqslant \ell_{n,t}$ (resp. to $\ell_{n,t} \leqslant \overline{\ell}_{n,t}$).

Note that the KKT conditions give that, at optimality:

$$\gamma_{n,t}(\ell_{n,t}, L_t) = \lambda_n + \underline{\mu}_{n,t} - \overline{\mu}_{n,t} \ , \tag{19}$$

where $\gamma_{n,t}(\ell_{n,t}, L_t) \triangleq c_t(L_t) + \ell_{n,t} c_t'(L_t)$ is the marginal cost of n. Let us consider $\boldsymbol{\ell}$ and $\hat{\boldsymbol{\ell}}$ two NEs. From (19), we get:

$$\begin{aligned} \ell_{n,t} < \overline{\ell}_{n,t} \Rightarrow \overline{\mu}_{n,t} = 0 \Rightarrow \gamma_{n,t}(\ell_{n,t}, L_t) \geqslant \lambda_n \\ \text{and } \ell_{n,t} > \underline{\ell}_{n,t} \Rightarrow \underline{\mu}_{n,t} = 0 \Rightarrow \gamma_{n,t}(\ell_{n,t}, L_t) \leqslant \lambda_n \end{aligned}$$

and the same inequalities hold for $\hat{\ell}$. First note that:

$$\left(\hat{\lambda}_n \leqslant \lambda_n \text{ and } \hat{L}_t \geqslant L_t\right) \Rightarrow \hat{\ell}_{n,t} \leqslant \ell_{n,t} \ , \tag{20}$$

$$\left(\hat{\lambda}_n \geqslant \lambda_n \text{ and } \hat{L}_t \leqslant L_t\right) \Rightarrow \hat{\ell}_{n,t} \geqslant \ell_{n,t} \ . \tag{21}$$

Let us show (20). If $\hat{\ell}_{n,t} = \underline{\ell}_{n,t}$ or $\ell_{n,t} = \overline{\ell}_{n,t}$, then $\hat{\ell}_{n,t} \leqslant \ell_{n,t}$ is clear. Else, $\hat{\ell}_{n,t} > \underline{\ell}_{n,t}$ and $\ell_{n,t} < \overline{\ell}_{n,t}$ so:

$$\gamma_{n,t}(\hat{\ell}_{n,t}, \hat{L}_t) \leqslant \hat{\lambda}_n \leqslant \lambda_n \leqslant \gamma_{n,t}(\ell_{n,t}, L_t) \leqslant \gamma_{n,t}(\ell_{n,t}, \hat{L}_t) \tag{22}$$

as $\gamma_{n,t}$ is increasing in L_t. As $c_t'(\hat{L}_t) > 0$ from Assumption 1, $\gamma_{n,t}$ is increasing in $\ell_{n,t}$ and we deduce that $\ell_{n,t} \geqslant \hat{\ell}_{n,t}$.

Now, let us consider $\mathcal{T}_1 = \{t \ : \ \hat{L}_t > L_t\}$ along with $\mathcal{T}_2 = \mathcal{T} \setminus \mathcal{T}_1 = \{t \ : \ \hat{L}_t \leqslant L_t\}$ and $\mathcal{N}_0 = \{n \ : \ \hat{\lambda}_n > \lambda_n\}$. Suppose $\mathcal{T}_1 \neq \emptyset$. From constraint (6a) and from (21), we have:

$$\forall n \in \mathcal{N}_0, \sum_{t\in\mathcal{T}_1} \hat{\ell}_{n,t} = E_n - \sum_{t\in\mathcal{T}_2} \hat{\ell}_{n,t} \leqslant E_n - \sum_{t\in\mathcal{T}_2} \ell_{n,t} = \sum_{t\in\mathcal{T}_1} \ell_{n,t} \ .$$

On the other hand, considering for $t \in \mathcal{T}_1$ and $n \notin \mathcal{N}_0$, we have from (20) that $\hat{\ell}_{n,t} \leqslant \ell_{n,t}$, and thus:

$$\sum_{t\in\mathcal{T}_1} \hat{L}_t = \sum_{t\in\mathcal{T}_1} \sum_{n\in\mathcal{N}_0} \hat{\ell}_{n,t} + \sum_{t\in\mathcal{T}_1} \sum_{n\notin\mathcal{N}_0} \hat{\ell}_{n,t} \leqslant \sum_{t\in\mathcal{T}_1} L_t \ , \tag{23}$$

which is in contradiction with the definition of $\mathcal{T}_1$. Thus $\mathcal{T}_1 = \emptyset$ and $\forall t,\ \hat{L}_t = L_t$. We can now precise (20) with:

$$\begin{aligned} &\left[\hat{\lambda}_n < \lambda_n \text{ and } \hat{L}_t = L_t\right] \Longrightarrow \\ &\quad \left[\hat{\ell}_{n,t} < \ell_{n,t} \text{ or } \hat{\ell}_{n,t} = \ell_{n,t} = \underline{\ell}_{n,t} \text{ or } \hat{\ell}_{n,t} = \ell_{n,t} = \overline{\ell}_{n,t}\right] \end{aligned} \tag{24}$$

and similarly for (21). Indeed, if $\hat{\ell}_{n,t} = \underline{\ell}_{n,t}$ (resp. if $\ell_{n,t} = \overline{\ell}_{n,t}$) then the implication holds because $\ell_{n,t} \geqslant \underline{\ell}_{n,t}$ (resp. $\ell_{n,t} \leqslant \overline{\ell}_{n,t}$). Else, $\hat{\ell}_{n,t} > \underline{\ell}_{n,t}$ and $\ell_{n,t} < \overline{\ell}_{n,t}$, and the same sequence of inequalities as in (22) gives $\gamma_{n,t}(\hat{\ell}_{n,t}, L_t) < \gamma_{n,t}(\ell_{n,t}, L_t)$, implying that $\hat{\ell}_{n,t} < \ell_{n,t}$.

Finally, suppose that there exists n s.t. $\hat{\lambda}_n < \lambda_n$. If only the two latter cases in (24) happen, then $\ell_{n,t} = \hat{\ell}_{n,t},\ \forall t$. Else, there is at least one t for which $\hat{\ell}_{n,t} < \ell_{n,t}$, so $E_n = \sum_t \hat{\ell}_{n,t} < \sum_t \ell_{n,t} = E_n$ which cannot happen. Thus, $\hat{\lambda}_n = \lambda_n$ for all n and (20) and (21) imply that $\ell_{n,t} = \hat{\ell}_{n,t}$ for all n and t.

Appendix B
Proof of Thm. 3: PoA upper bound

The proof relies on the notion of *local smoothness* introduced in [34]. The idea is to get a tighter bound than [34] by specifying the parameters of the affine price functions $(c_t)_t$ and by using the upper bound on L_t instead of looking at the worst possible cases as done in [34].

Let $\kappa_t \triangleq \alpha_t/\beta_t$ so that $c_t(L) = \beta_t(L+\kappa)$. From [34], we know that if there exist $\lambda, \mu > 0$ and a profile $\hat{\boldsymbol{\ell}} \in \mathcal{L}$ satisfying for each $t \in \mathcal{T}$ and each $\boldsymbol{\ell} \in \mathcal{L}$:

$$\hat{L}_t(L_t + \kappa_t) + \frac{\hat{L}_t^2}{4} \leqslant \lambda \hat{L}_t(\hat{L}_t + \kappa_t) + \mu L_t(L_t + \kappa_t), \tag{25}$$

where $\hat{L}_t = \sum_n \hat{\ell}_{n,t}$ and $L_t = \sum_n x_{n,t}$, then $\mathcal{G}$ is locally λ, μ-smooth for $\hat{\boldsymbol{\ell}}$, *i.e.* for any admissible profile $\boldsymbol{\ell} \in \mathcal{L}$:

$$\sum_{n\in\mathcal{N}} b_n(\boldsymbol{\ell}) + \nabla_n b_n(\boldsymbol{\ell})^T(\hat{\boldsymbol{\ell}}_n - \boldsymbol{\ell}_n) \leqslant \lambda \mathrm{SC}(\hat{\boldsymbol{\ell}}) + \mu \mathrm{SC}(\boldsymbol{\ell}) \ ,$$

where $\mathrm{SC}(\boldsymbol{\ell}) = \sum_n b_n(\boldsymbol{\ell})$. In that case, it follows from [34] that the PoA is bounded by $\lambda/(1-\mu)$. We want to find the best possible λ, μ such that (25) holds for each $t \in \mathcal{T}$. For the remaining of the proof, we fix t and omit subscript t in the notations. As done in [34], we introduce:

$$\phi_{xy}(\mu) \triangleq \frac{y(x+\kappa) + \frac{y^2}{4} - \mu x(x+\kappa)}{y(y+\kappa)}, \ \lambda^*(\mu) \triangleq \sup_{x,y\geqslant 0} \phi_{xy}(\mu) \ .$$

$\lambda^*(\mu)$ is the minimum value of $\lambda > 0$ such that (25) holds with values (λ, μ). Let us compute an explicit expression of $\lambda^*(\mu)$. If $x = 0$, $\phi_{0,y}(\mu) = \frac{y+4b}{4(y+\kappa)}$ and $\frac{\partial \phi_{0,y}}{\partial y} < 0$ so $\sup_{x,y} \phi_{x,y}$ would be attained with $y = 0$ and is $\phi_{0,0} = 1$. Otherwise:

$$0 = \frac{\partial \phi}{\partial x} \Leftrightarrow \frac{1}{y(y+\kappa)}(y - 2\mu x - \mu\kappa) \Rightarrow x = \frac{y - \kappa\mu}{2\mu}$$

but as $x \geqslant 0$, this supposes that $y \geqslant \mu\kappa$. We compute:

$$\phi_{\frac{y-\kappa\mu}{2\mu}, y} = \frac{1}{y(y+\kappa)4\mu}\left((y+\kappa\mu)^2 + \mu y^2\right) \triangleq h(y) \ .$$

We can see that h' vanishes on $\mathbb{R}_+$ at $y_+ \triangleq \frac{\kappa\mu^2 + \kappa\mu\sqrt{\mu^2+1-\mu}}{1-\mu}$ that gives a min of h so h is decreasing then increasing. At the lower bound $y=\kappa\mu$, we get $\phi=\frac{\kappa\mu+4b}{4(\kappa\mu+\kappa)}=\frac{\mu+4}{4(\mu+1)}=\frac{1}{4}+\frac{3}{4(\mu+1)}<1$ which is not max as $\phi_{0,0}=1$. At the upper bound $y=\overline{L}$, we have $h(\overline{L})=\frac{(\overline{L}+\kappa\mu)^2+\mu\overline{L}^2}{\overline{L}(\overline{L}+\kappa)4\mu}=\lambda^*(\mu)$. Last, to compute the best bound $\inf_\mu \lambda^*(\mu)/(1-\mu)$, let us consider:

$$g(\mu)\triangleq 4\overline{L}(\overline{L}+\kappa)\frac{\lambda^*(\mu)}{1-\mu}=\frac{(\overline{L}+\kappa\mu)^2+\mu\overline{L}^2}{\mu(1-\mu)} \ .$$

If we denote $\varphi\triangleq(1+r)^2$ and $r\triangleq\kappa/\overline{L}$, $g(\mu)$ is minimal at $\mu^*\triangleq(-1+\sqrt{1+\varphi})/\varphi$. We finally get our PoA bound as:

$$\frac{\lambda^*(\mu^*)}{1-\mu^*}=\frac{(3+2r)+2\sqrt{1+\varphi}}{4(1+r)}=\frac{1}{2}\left(1+\sqrt{1+\frac{1}{\varphi}}+\frac{1}{2\sqrt{\varphi}}\right)$$
$$=\tfrac{1}{2}\left(1+\sqrt{1+(1+r)^{-2}}+(2(1+r))^{-1}\right)\leqslant 1+\tfrac{3}{4(1+r)} \ .$$

The last inequality gives a more explicit bound and is obtained from $\sqrt{a^2+b^2}\leqslant a+b$ valid for any $a,b\geqslant 0$.
Next, following [34], for $\boldsymbol{\ell},\boldsymbol{\ell}'\in\mathcal{L}$ (admissible solutions):

$$\begin{aligned}
&\textstyle\sum_n b_n(\boldsymbol{\ell})+\nabla_n b_n(\boldsymbol{\ell})^T(\boldsymbol{\ell}'-\boldsymbol{\ell})\\
&=\textstyle\sum_n\sum_{t\in\mathcal{T}}\ell_{n,t}\cdot c_t(L_t)+(\ell'_{n,t}-\ell_{n,t})\left(c_t(L_t)+\ell_{n,t}c'_t(L_t)\right)\\
&=\textstyle\sum_t L'_t\cdot c_t(L_t)+c'_t(L_t)\sum_n\left(\ell'_{n,t}\ell_{n,t}-{\ell_{n,t}}^2\right)\\
&\leqslant\textstyle\sum_t L'_t\cdot c_t(L_t)+c'_t(L_t)\cdot\frac{{L_t}^2}{4}\\
&=\textstyle\sum_t\beta_t\left[L'_t(L_t+\kappa_t)+\frac{{L_t}^2}{4}\right] &(26)\\
&\leqslant\textstyle\sum_t\beta_t\left[\lambda L'_t(L'_t+\kappa_t)+\mu L_t(L_t+\kappa_t)\right] &(27)\\
&=\lambda\mathrm{SC}(\boldsymbol{\ell}')+\mu\mathrm{SC}(\boldsymbol{\ell}) \ ,
\end{aligned}$$

where (27) is valid if (λ,μ) is chosen such that:

$$\forall t\in\mathcal{T},\ \lambda\geqslant\frac{(\overline{L}_t+\kappa_t\mu)^2+\mu\overline{L}_t^2}{\overline{L}_t(\overline{L}_t+\kappa_t)4\mu}\triangleq\lambda^*_{\kappa_t}(\mu) \ .$$

Let us denote $t_0\triangleq\underset{t}{\text{argmin}}\ \kappa_t$ and choose $\mu^*\triangleq\mu^*_{t_0}$, $\lambda^*\triangleq\lambda^*_{\kappa_{t_0}}(\mu^*)$ (the optimal (λ,μ) for t_0), then we have to check that for all $t\in\mathcal{T}$, $\lambda^*\geqslant\lambda^*_{\kappa_t}(\mu^*)$. For that, it is sufficient to show that $r\mapsto\lambda^*_r(\mu^*)$ is decreasing on $[r_{t_0},r_t]$, which is true if $r_t<-1+\sqrt{1+\frac{1-\mu^*}{\mu^{*2}}}\iff\varphi_{r_t}<\varphi_{r_{t_0}}+2+\sqrt{1+\varphi_{r_{t_0}}}$ with $\varphi_r=(1+r)^2$, which gives condition (9) stated in Thm. 3.

APPENDIX C
PROOF OF PROP. 1: STRONG STABILITY OF $\mathcal{G}$

We denote by $G(\boldsymbol{\ell})\triangleq J_F(\boldsymbol{\ell})$ the Jacobian of operator $F=(\nabla_n b_n)_{n\in\mathcal{N}}$. Since functions b_n are twice differentiable, condition (13) is equivalent to having the matrix $G(\boldsymbol{\ell})+G^T(\boldsymbol{\ell})$ positive definite for all $\boldsymbol{\ell}\in\mathcal{L}$, that is, $G(\boldsymbol{\ell})+G(\boldsymbol{\ell})^T\succ 0$.

As $b_n=\sum_t b_{n,t}$, with $b_{n,t}(\boldsymbol{\ell}_t)\triangleq\ell_{n,t}c_t(L_t)$, is separable in t, we can re-index the matrix $G(\boldsymbol{\ell})$ to have a diagonal block hourly matrix $G(\boldsymbol{\ell})=\mathrm{diag}(G_1,...G_T)$, with $G_t(\boldsymbol{\ell}_t)\triangleq\left(\frac{\partial^2 b_{n,t}}{\partial\ell_{n,t}\partial\ell_{m,t}}(\boldsymbol{\ell}_t)\right)_{n,m\in\mathcal{N}^2}$ and we get for all t:

$$G_t(\boldsymbol{\ell}_t)+G_t(\boldsymbol{\ell}_t)^T=\left(\frac{\partial^2 b_{n,t}(\boldsymbol{\ell}_t)}{\partial\ell_{m,t}\partial\ell_{n,t}}+\frac{\partial^2 b_{m,t}(\boldsymbol{\ell}_t)}{\partial\ell_{n,t}\partial\ell_{m,t}}\right)_{n,m} .$$

Let $t\in\mathcal{T}$ and $\boldsymbol{x}\in\mathbb{R}^N\setminus\{0\}$. Furthermore, let $\sigma(\boldsymbol{x},\boldsymbol{\ell})\triangleq\boldsymbol{x}^T\left(G_t(\boldsymbol{\ell}_t)+G_t^T(\boldsymbol{\ell}_t)\right)\boldsymbol{x}$. For notation simplicity, let us forget the index t and the argument (L) in functions c_t. We have:

$$\sigma(\boldsymbol{x},\boldsymbol{\ell})=\sum_{n=1}^{N}2x_n^2(\ell_n c''+2c')+2\sum_{n<m}x_nx_m\left((\ell_n+\ell_m)c''+2c'\right)=$$
$$\sum_{n=1}^{N}2x_n^2\left(r_n\gamma+(1-r_n)a\right)+2\sum_{n<m}x_nx_m\left((r_n+r_m)\gamma+(1-r_n-r_m)a\right)$$

with $r_n\triangleq\ell_n/L$, $a=2c'(L)$ and $\gamma\triangleq 2c'(L)+Lc''(L)$. Then:

$$\sigma=a\sum_n x_n^2+a\Big(\sum_n\big(1-r_n\left(1-\tfrac{\gamma}{a}\right)\big)x_n\Big)^2-\tfrac{(a-\gamma)^2}{a}\sum_{n,m}r_nr_mx_nx_m$$

which is the sum of three quadratic forms: $q_1(\boldsymbol{x})=a\sum x_n^2$ which has one eigenvalue a of multiplicity N, $q_2(\boldsymbol{x})=a\left(\boldsymbol{x^Tv^Tvx}\right)$ with $v_n\triangleq\sum_n 1-\frac{\ell_n}{L}\left(1-\frac{\gamma}{a}\right)$ of rank one whose nonzero eigenvalue is $a||\boldsymbol{v}||_2^2$, and a negative form of rank one $q_3(\boldsymbol{x})=-\frac{1}{a}(a-\gamma)^2\left(\sum_{n,m}\frac{\ell_n}{L}\frac{\ell_m}{L}x_nx_m\right)$ whose nonzero eigenvalue is $-\frac{1}{a}(a-\gamma)^2\sum_n\left(\frac{\ell_n}{L}\right)^2$.

We deduce that the quadratic form q_1+q_2 is positive definite, and that its eigenvalues are a with multiplicity $N-1$ and $a(1+||\boldsymbol{v}||_2^2)$ with multiplicity 1. Next, we use the following result from perturbation theory:

Theorem 7 (Horn and Johnson, 2012, [49, p367]). *Let $A,E\in\mathcal{M}_n$ be two Hermitian matrices and let $\lambda_1^M\leqslant...\leqslant\lambda_n^M$ denotes the (real) ordered eigenvalues of an Hermitian matrix M. Then we the following inequalities hold:*

$$\forall k=1\dots n,\ \lambda_1^E\leqslant\lambda_k^{A+E}-\lambda_k^A\leqslant\lambda_n^E$$
$$\textit{and}\ \left|\lambda_k^{A+E}-\lambda_k^A\right|\leqslant\rho(E)=|||E|||_2 \ .$$

Applying this theorem with $A=q_1+q_2$ and perturbation $E=q_3$ we get that the smallest eigenvalue λ_1^{A+E} of σ verifies:

$$\lambda_1^{A+E}\geqslant\min\{\mathrm{Sp}(q_1+q_2)\}\ -\tfrac{(a-\gamma)^2}{a}\sum_n r_n^2$$
$$=a\Big(1-\big(1-\tfrac{\gamma}{a}\big)^2\textstyle\sum_n r_n^2\Big).$$

Replacing a and γ, we can get the condition of Assumption 2.

APPENDIX D
PROOF OF THM. 4: CONVERGENCE OF CBRD

The key of the proof is that, under Assumption 3, the game is an exact potential game [43] with convex potential:

$$\Phi(\boldsymbol{\ell})=\textstyle\sum_{t\in\mathcal{T}}\alpha_tL_t+\frac{\beta_t}{2}(L_t^2+\sum_n\ell_{n,t}^2) \ ,$$

that is, for any $\boldsymbol{\ell}\in\mathcal{L}$ and any n, $\nabla_n\Phi(\boldsymbol{\ell})=\nabla_nb_n(\boldsymbol{\ell})$. Thus, the NE corresponds to the minimum of Φ and we have, for any $\boldsymbol{\ell}\in\mathcal{L}$, $\underset{\boldsymbol{\ell}_n\in\mathcal{L}_n}{\text{argmin}}\ b_n(\boldsymbol{\ell}_n,\boldsymbol{\ell}_{-n})=\underset{\boldsymbol{\ell}_n\in\mathcal{L}_n}{\text{argmin}}\ \Phi(\boldsymbol{\ell}_n,\boldsymbol{\ell}_{-n})$. Therefore, running Algo 1 is equivalent to performing an alternating block coordinate minimization on Φ. According to [44, Thm. 6.1]:

$$\Phi(\boldsymbol{\ell}^{(k)})-\Phi(\boldsymbol{\ell}^{\mathrm{NE}})\leqslant\tfrac{1}{k}\times 2MN^2R^2\Omega \ , \qquad (28)$$

with $M=\max_n M_n=2\max_t\beta_t$ (max of Lipschitz constants of $\nabla_nb_n=\nabla_n\Phi$), $R=\max_{\boldsymbol{\ell}}\{\left\|\boldsymbol{\ell}-\boldsymbol{\ell}^{\mathrm{NE}}\right\|\ ;\Phi(\boldsymbol{\ell})\leqslant\Phi(\boldsymbol{\ell}^{(0)})\}$

and $\Omega = \max\{\frac{2}{MN^2R^2} - 2,\ \Phi(\boldsymbol{\ell}^{(1)}) - \Phi(\boldsymbol{\ell}^{\mathrm{NE}}),\ 2\}$. But Φ is also strongly convex, that is, for any $\boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L}$:

$$\Phi(\boldsymbol{\ell}) - \Phi(\boldsymbol{\ell}') \geqslant \langle \nabla\Phi(\boldsymbol{\ell}'), \boldsymbol{\ell} - \boldsymbol{\ell}' \rangle + \tfrac{a}{2} \|\boldsymbol{\ell} - \boldsymbol{\ell}'\|^2 \qquad (29)$$

with $a = 2\min_t \beta_t$. Also, the minimality of $\boldsymbol{\ell}^{\mathrm{NE}}$ on the convex set $\mathcal{L}$ implies that for any $\boldsymbol{\ell} \in \mathcal{L}$, $\langle \nabla\Phi(\boldsymbol{\ell}^{\mathrm{NE}}), \boldsymbol{\ell} - \boldsymbol{\ell}^{\mathrm{NE}} \rangle \geqslant 0$. Then from (29) , we get for any $k \geqslant 0$:

$$\begin{aligned} \frac{a}{2} \left\| \boldsymbol{\ell}^{(k)} - \boldsymbol{\ell}^{\mathrm{NE}} \right\|^2 &\leqslant \Phi(\boldsymbol{\ell}^{(k)}) - \Phi(\boldsymbol{\ell}^{\mathrm{NE}}) + \langle \nabla\Phi(\boldsymbol{\ell}^{\mathrm{NE}}), \boldsymbol{\ell}^{\mathrm{NE}} - \boldsymbol{\ell}^{(k)} \rangle \\ &\leqslant \Phi(\boldsymbol{\ell}^{(k)}) - \Phi(\boldsymbol{\ell}^{\mathrm{NE}}) \ , \end{aligned}$$

and from (28) we get the convergence result of Thm. 4.

Appendix E
Proof of Thm. 5: Convergence of SIRD

We analyze the convergence of the sequence $(\mathrm{T}_\gamma^k(\boldsymbol{\ell}))_k$ where $[\mathrm{T}_\gamma(\boldsymbol{\ell})]_n \triangleq \Pi_{\mathcal{L}_n}(\boldsymbol{\ell}_n - \gamma\nabla_n b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}))$. First notice that the unique NE of the game $\boldsymbol{\ell}^{\mathrm{NE}}$ is the unique fixed point of T_γ i.e. $\boldsymbol{\ell}^{\mathrm{NE}} = \mathrm{T}_\gamma(\boldsymbol{\ell}^{\mathrm{NE}})$. The idea is then to prove that T_γ is a η-contraction, for a given norm $\|\cdot\|$ which will imply the convergence rate $\|T^k(\boldsymbol{\ell}^{(0)}) - \boldsymbol{\ell}^{\mathrm{NE}}\| \leqslant \eta^k \|\boldsymbol{\ell}^{(0)} - \boldsymbol{\ell}^{\mathrm{NE}}\|$, for any initial condition $\boldsymbol{\ell}^{(0)} \in \mathbb{R}^{N\times T}$.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ for any positive integer d. As the projection on a convex set is nonexpansive [50, Corollary 12.20], we get for $\boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L}$:

$$\begin{aligned} &\|\mathrm{T}_\gamma(\boldsymbol{\ell}) - \mathrm{T}_\gamma(\boldsymbol{\ell}')\|^2 = \textstyle\sum_{n=1}^N \|[\mathrm{T}_\gamma(\boldsymbol{\ell})]_n - [\mathrm{T}_\gamma(\boldsymbol{\ell}')]_n\|^2 \\ &= \textstyle\sum_{n=1}^N \|\Pi_{\mathcal{L}_n}(\boldsymbol{\ell}_n - \gamma\nabla_n b_n(\boldsymbol{\ell})) - \Pi_{\mathcal{L}_n}(\boldsymbol{\ell}'_n - \gamma\nabla_n b_n(\boldsymbol{\ell}'))\|^2 \\ &\leqslant \textstyle\sum_{n=1}^N \|\boldsymbol{\ell}_n - \boldsymbol{\ell}'_n + \gamma(\nabla_n b_n(\boldsymbol{\ell}') - \nabla_n b_n(\boldsymbol{\ell}))\|^2 \\ &= \textstyle\sum_{n=1}^N \|\boldsymbol{\ell}_n - \boldsymbol{\ell}'_n\|^2 + \gamma^2 \|\nabla_n b_n(\boldsymbol{\ell}) - \nabla_n b_n(\boldsymbol{\ell}')\|^2 \\ &- 2\gamma \langle \nabla_n b_n(\boldsymbol{\ell}) - \nabla_n b_n(\boldsymbol{\ell}'), \boldsymbol{\ell}_n - \boldsymbol{\ell}'_n \rangle \ . \end{aligned}$$

Since $\nabla_n b_n$ is M_n-Lipschitz for each n, we have $\sum_{n=1}^N |\nabla_n b_n(\boldsymbol{\ell}) - \nabla_n b_n(\boldsymbol{\ell}')|^2 \leqslant NM^2 \|\boldsymbol{\ell} - \boldsymbol{\ell}'\|^2$ where $M \triangleq \max_n M_n$. From a-strong stability (Def. 3), we get:

$$\|\mathrm{T}_\gamma(\boldsymbol{\ell}) - \mathrm{T}_\gamma(\boldsymbol{\ell}')\|^2 \leqslant \eta \|\boldsymbol{\ell} - \boldsymbol{\ell}'\|^2 \ ,$$

with $\eta \triangleq 1 + NM^2\gamma^2 - 2\gamma\alpha$. Minimizing on $\gamma > 0$ gives $\gamma = \frac{\alpha}{NM^2}$ and $\eta = 1 - \frac{\alpha^2}{NM^2} < 1$, and T_γ is a contraction.

Appendix F
Proof of Thm. 6: consistency of DR Procedure

First, the NE is unique in any "subgame" $\mathcal{G}^{(t)}$ played on the subset $\mathcal{T}^{(t)} = \{t, \dots, \mathcal{T}\}$ (considered at t in the procedure). In the case where we assume that the operator F (*cf* Def. 3) is strictly monotone on $\mathcal{L}$, then the operator $F^{(t)} : \boldsymbol{\ell}^{(t)} \mapsto [\nabla_{\boldsymbol{\ell}_n^{(t)}} b_n(\boldsymbol{\ell}^{(t)})]_{n\in\mathcal{N}}$ restricted to the set $\mathcal{T}^{(t)}$, is also strictly monotone on $\mathcal{L}^{(t)} = \prod_n \mathcal{L}_n^{(t)}$. In the case where we consider Assumption 1 and that the sets $(\mathcal{L}_n)_n$ have the structure (6), this structure is inherited for the sets $(\mathcal{L}_n^{(t)})_n$ so Thm. 2 can be applied on the game $\mathcal{G}^{(t)}$ to ensure the uniqueness.

Let $t_0 \in \{1, \dots, T-1\}$ and $\mathcal{G}^{(t_0)}$ the DR-game played at t_0. Let $\boldsymbol{\ell}^{(t_0)}$ be the unique NE of $\mathcal{G}^{(t_0)}$. From the variational inequality characterization of an NE, we have:

$$\langle F^{(t_0)}(\boldsymbol{\ell}^{(t_0)}), \boldsymbol{\lambda}^{(t_0)} - \boldsymbol{\ell}^{(t_0)} \rangle \geqslant 0,\ \forall \boldsymbol{\lambda}^{(t_0)} \in \mathcal{L}^{(t_0)} \ . \qquad (30)$$

Let $\mathcal{G}^{(t_0+1)}$ the DR-game on hours $\{t_0+1, \dots T\}$ with updated strategy sets $\mathcal{L}_n^{(t_0+1)} \triangleq \{(\ell_{n,s})_{s>t_0} | (\ell_{n,t_0}^{(t_0)}, [\ell_{n,s}]_{s>t_0}) \in \mathcal{L}_n^{(t_0)}\}$ for each n. Let $\boldsymbol{\lambda}^{(t_0+1)} \in \mathcal{L}^{(t_0+1)}$, then $\boldsymbol{\lambda}^{(t_0)} \triangleq (\boldsymbol{\ell}_{t_0}^{(t_0)}, \boldsymbol{\lambda}^{(t_0+1)}) \in \mathcal{L}^{(t_0)}$.

$$\begin{aligned} 0 &\leqslant \langle F^{(t_0)}(\boldsymbol{\ell}^{(t_0)}), \boldsymbol{\lambda}^{(t_0)} - \boldsymbol{\ell}^{(t_0)} \rangle \\ &= 0 + \langle F^{(t_0+1)}((\boldsymbol{\ell}_s^{(t_0)})_{s>t_0}), \boldsymbol{\lambda}^{(t_0+1)} - (\boldsymbol{\ell}_s^{(t_0)})_{s>t_0} \rangle \ , \end{aligned}$$

which shows, from (30), that $(\boldsymbol{\ell}_s^{(t_0)})_{s>t_0}$ is an NE of the game $\mathcal{G}^{(t_0+1)}$. From the uniqueness of the NE in $\mathcal{G}^{(t_0+1)}$, we finally conclude that $(\boldsymbol{\ell}_s^{(t_0)})_{s>t_0} = \boldsymbol{\ell}^{(t_0+1)}$.

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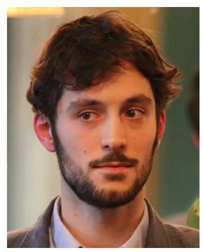

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